ISBN: 978-3-95450-240-0 ISSN: 2673-5539 doi:10.18429/JACoW-RuPAC2021-TUPSB13

CHARGED PARTICLE DYNAMICS OPTIMIZATION IN DISCRETE SYSTEMS

E. D. Kotina†, D. A. Ovsyannikov, St. Petersburg State University, St. Petersburg, Russia

Abstract

Discrete optimization methods of dynamic systems are widely presented in the scientific literature. However, to solve various problems of beam dynamics optimization, it is necessary to create special optimization models that would take into account specifics of the problems under study. The paper proposes a new mathematical model that includes joint optimization of a selected (calculated) motion and an ensemble of perturbed motions. Functionals of a general form are considered, which make it possible to estimate various characteristics of a charged particle beam and the dynamics of the calculated trajectory. The optimization of a bundle of smooth and nonsmooth functionals is investigated. These functionals estimate both integral characteristics of the beam as a whole and various maximum deviations of the parameters of the particle beam. The variation of a bundle of functionals is given in an analytical form, that allows us to construct directed optimization methods. The selected trajectory can be taken, for example, as the trajectory of a synchronous particle or the center of gravity of a beam (closed orbit). We come to discrete models when we consider the dynamics of particles using transfer matrices or transfer maps. Optimization problems can be of orbit correction, dynamic aperture optimization, and many other optimization problems in both cyclic and linear accelerators of charged particle beams.

INTRODUCTION

Discrete systems are becoming increasingly important in theory and practical application in optimal control and optimization problems [1-5]. This is due to the fact that many problems are described by discrete equations, since in practice information about the state of the process comes discretely, and control of the dynamic process is implemented most often at discrete moments of time. The standard approach to the design of various control systems involves the initial calculation of the selected motion and the subsequent study of perturbed motions using equations in deviations. This approach, however, does not always lead to the desired results. When analyzing perturbed motions, it turns out that their dynamic characteristics are not always satisfactory from one point of view or another. This is a consequence of the significant dependence of the perturbed motions on the selected motion. In this paper, a mathematical model is proposed that allows simultaneous optimization of the selected motion and the ensemble of perturbed motions in discrete systems. In this case, the simultaneous optimization of smooth and nonsmooth functionals is considered.

The classical formulations of optimal control problems in discrete systems are quite well known and studied [1]. These problems can be considered as tasks of single trajectories control. Along with them, non-standard problems of the theory of optimal control are being developed. In particular, the control problems of ensembles of trajectories (beams) were considered under various cost functionals in continuous and discrete-time systems [2]. Further, non-standard problems of joint optimization of program motion and perturbed motions in continuous systems [6, 7], as well as in discrete ones [3-5] were developed. The problems of simultaneous optimization of smooth and nonsmooth functionals defined on a program motion and a beam of perturbed trajectories in continuous and discrete systems were further developed [8-10].

This article is devoted to the construction of new methods for optimizing the bundle of smooth and nonsmooth functionals in discrete systems.

PROBLEM STATEMENT

In this paper a dynamic system is described by the discrete equations of the following type

$$x(k+1) = f\big(k, x(k), u(k)\big), \quad (1)$$
$$y(k+1) = F\big(k, x(k), y(k), u(k)\big), \quad (2)$$
$$k = 0, \dots, N-1,$$

where $x(k) - n -$ dimensional phase vector, characterizing the state of the system, $y(k) - m -$ dimensional phase vector, $u(k) - r -$ dimensional vector, $f(k) = f\big(k, x(k), u(k)\big)$ is $n-$dimensional vector function, $F(k) = F\big(k, x(k), y(k), u(k)\big) - m -$ dimensional vector function. We suppose that $f(k)$ is defined and continuous in $\Omega_x \times U(k)$ by the arguments $\big(x(k), u(k)\big)$ for $k = \overline{1, N}$ along with their partial derivatives. We also assume that $F(k)$ is defined and continuous in $\Omega_x \times \Omega_y \times U(k)$ by the arguments $\big(x(k), y(k), u(k)\big)$ for $k = \overline{1, N}$ along with their first and second partial derivatives. Here $\Omega_x \subset R^n$, $\Omega_y \subset R^m$, $U(k)$ – a compact set in R^r, $k = \overline{1, N}$. We suppose that for a given vector $u(k)$, the vector $x(k)$ and the vector $y(k)$ uniquely determine the phase state $y(k+1)$ of the perturbed particle at the $k-$th step and vice versa, by $y(k+1) -$ the state of the perturbed particle at the previous step.

Equation (1) describes the dynamics of the selected motion. Equation (2) describes the perturbed motion.

We assume, that $x(0) = x_0$, ($x_0 \in \Omega_x \subset R^n$) and the initial state of the system (2) is described by set $M_0 -$ a compact set of nonzero measure in R^m, the sequence of vectors $\{u(0), u(1), \dots, u(N-1)\}$ we will call control and denote u for brevity, $x = \{x(0), x(1), \dots, x(N)\} =$

† e.kotina@spbu.ru

ISBN: 978-3-95450-240-0 ISSN: 2673-5539 doi:10.18429/JACoW-RuPAC2021-TUPSB13

$x(x_0, u)$ – the trajectory of selected motion corresponding to the control u and the initial state x_0, $x(k) = x(k, x_0, u)$ – the state of the system at the step k, $y = \{y(0), y(1), \dots, y(N)\} = y(x, y_0, u)$ – the trajectory of perturbed motion, $y(k) = y(k, x, y_0, u)$ – the state of the system at the step k. The set of trajectories $y(x, y_0, u)$ corresponding to the initial state x_0, the control u and different initial states $y_0 \in M_0$ we will call a beam of trajectories or simply the beam. State of the beam at the k –th step is the cross-section of trajectories ensemble denoted as $M_{k,u} = \{y(k): y(k) = y(k, y_0, x, u), y_0 \in M_0\}$. The controls satisfying conditions $u(k) \in U(k)$, $k = \overline{1, N-1}$ we call admissible.

On the trajectories of the system (1)-(2), we introduce cost functionals that allow us to evaluate the dynamics of the calculated and perturbed motion and to carry out their joint optimization:

$$I_1(u) = \sum_{k=1}^{N-1} g\big(k, x(k)\big) + g(N, x(N)), \tag{3}$$

$$I_2(u) = \max_{y(N)\in M_{N,u}} \Phi\big(y(N)\big), \tag{4}$$

$$I(u) = I_1(u) + I_2(u). \tag{5}$$

Here $g_k = g\big(k, x(k)\big)$, $k = 0, 1, \dots, N-1$ and $g_N = g(N, x(N))$ continuously differentiable functions by x, $\Phi(y(N))$ – a non-negative continuously differentiable function with respect to y.

We consider the problem of functional (5) minimization for all admissible controls.

FUNCTIONAL VARIATION

Let u – admissible control. We will consider the following variation of this control:

$$\tilde{u}_\varepsilon = u + \varepsilon \Delta u. \tag{6}$$

We suppose, that $\tilde{u}_\varepsilon$ is an admissible control, when $\varepsilon \epsilon [0, \bar{\varepsilon})$, $\bar{\varepsilon} > 0$, in this case, we will call Δu the permissible direction of the control variation.

The trajectory increment at the k –th step $\Delta x(k) = \tilde{x}(k, x_0, \tilde{u}_\varepsilon) - x(k, x_0, u)$ and the trajectory increment of perturbed motion at the k –th step $\Delta y(k) = \tilde{y}(\tilde{x}, y_0, \tilde{u}_\varepsilon) - y(x, y_0, u)$, for control in Eq. (6), can be represented as [1]

$$\Delta x(k+1) = \varepsilon \delta x(k+1) + o(\varepsilon),$$
$$\Delta y(k+1) = \varepsilon \delta y(k+1) + o(\varepsilon),$$

where $k = \overline{1, N}$ and variations δx и δy satisfy the equations [10]:

$$\delta x(k+1) = \frac{\partial f(k)}{\partial x(k)} \delta x(k) + \frac{\partial f(k)}{\partial u(k)} \Delta u(k), \tag{7}$$

$$\delta y(k+1) = \frac{\partial F(k)}{\partial x(k)} \delta x(k) + \frac{\partial F(k)}{\partial y(k)} \delta y(k) + \frac{\partial F(k)}{\partial u(k)} \Delta u(k), \tag{8}$$

$$k = 0, \dots, N-1.$$

At the same time, the initial conditions take place $\delta x(0) = 0$ and $\delta y(0) = 0$.

The increment of the functional in Eq. (5) can be written as

$$I(\tilde{u}_\varepsilon) - I(u) = \varepsilon \delta I + o(\varepsilon) = \varepsilon \delta I_1 + \varepsilon \delta I_2 + o(\varepsilon), \tag{9}$$

where δI_1 – functional (3) variation, δI_2 – functional (4) variation, δI – functional (5) variation.

Variation of the Functional $I_1(u)$

We write out a variation of the functional (3) using Eq. (7). We get

$$\delta I_1 = \sum_{k=1}^{N-1} \frac{\partial g_k}{\partial x_k} \delta x_k + \frac{\partial g_N}{\partial x_N} \delta x_N.$$

Variation δI_1 can be converted to the form

$$\delta I_1 = \sum_{k=0}^{N-1} \left[\beta^T(k+1) \frac{\partial f(k, x(k), u(k))}{\partial u(k)}\right] \Delta u(k), \tag{10}$$

using the following auxiliary vector functions

$$\beta^T(N) = \frac{\partial g_N}{\partial x_N},$$
$$\beta^T(k) = \beta^T(k+1) \frac{\partial f(k)}{\partial x(k)} + \frac{\partial g_k}{\partial x(k)}, \tag{11}$$
$$k = 1, \dots, N-1.$$

Variation of the Functional $I_2(u)$

We write down a variation of the functional $I_2(u)$

$$\delta I_2 = \max_{y_0 \in Y_0(u)} \left\{\frac{\partial \Phi(y_N)}{\partial y_N} \delta y_N\right\},$$

where $Y_0(u)$ is defined as follows [10]:

$$Y_0(u) = \Big\{y_0 \in M_0: \Phi\big(y(N)\big) = \Phi(y(N, x(x_0, u), y_0, u)) = \max_{y_N \in M_{N,u}} \Phi(y_N)\Big\}.$$

The variation of the functional $I_2(u)$, using equations (8), can be converted to the form

$$\delta I_2 = \max_{y_0 \in Y_0(u)} \left\{\sum_{k=0}^{N-1} \left[\xi^T(k+1) \frac{\partial f\big(k, x(k), u(k)\big)}{\partial u(k)} + \gamma^T(k+1) \frac{\partial F(k, x(k), y(k), u(k))}{\partial u(k)}\right] \Delta u(k)\right\}, \tag{12}$$

using the following vector functions

$$\xi^T(N) = 0, \quad \gamma^T(N) = \frac{\partial \Phi(y_N)}{\partial y_N},$$
$$\xi^T(k) = \gamma^T(k+1) \frac{\partial F(k)}{\partial x(k)} + \xi^T(k+1) \frac{\partial f(k)}{\partial x(k)}, \tag{13}$$

ISBN: 978-3-95450-240-0 ISSN: 2673-5539 doi:10.18429/JACoW-RuPAC2021-TUPSB13

$$\gamma^T(k) = \gamma^T(k+1)\frac{\partial F(k)}{\partial y(k)},$$
$$k = 1, \dots, N-1.$$

Variation of the Functional $I(u)$

Let us consider the functional $I(u)$. It follows from Eq. (9) and Eq. (10), Eq. (12) that the variation of the functional $I(u)$, using auxiliary functions (11), (13), can be represented as

$$\delta I_2 = \max_{y_0 \in Y_0(u)} \left\{ \sum_{k=0}^{N-1} \left[\xi^T(k+1)\frac{\partial f(k,x(k),u(k))}{\partial u(k)} + \beta^T(k+1)\frac{\partial f(k,x(k),u(k))}{\partial u(k)} + \gamma^T(+1)\frac{\partial F(k,x(k),y(k),u(k))}{\partial u(k)} \right] \Delta u(k) \right\}. \quad (14)$$

Thus, expression (14) gives us an analytical representation of the variation of the functional. On the basis of this representation, it is possible to build effective optimization methods taking into account the dynamics of program motion and perturbed motions. This may be important when optimizing the dynamics of charged particles in linear and cyclic accelerators.

CONCLUSION

When optimizing only the functional $I_1(u)$, the control choice does not depend on the perturbed movements, and they do not affect this choice. When optimizing the functional $I_2(u)$ we see from Eq. (12) that the controls depend on both the selected motion and the perturbed motions, and only the "worst" particles are taken into account in the perturbed motions. When studying the functional $I(u)$**,** equal to the sum of the functionals $I_1(u)$ and $I_2(u)$, we simultaneously optimize both the dynamics of the selected motion and the dynamics of perturbed motions, namely, we estimate the dynamics of the "worst particles", i.e. the greatest deviations from the selected motion. This approach can be effective in solving various problems of optimizing the dynamics of charged particles in accelerators and beam formation systems [8, 9, 11]. We especially note the problems and the need to optimize the dynamic aperture (DA) in cyclic accelerators and colliders [12, 13]. The proposed approach makes it possible to construct new optimization methods. Note that DA is understood as the deflection (in the x or y plane) of the particle that is most distant from the center of the beam (from the trajectory of the central particle), but at the same time remains within the aperture of the accelerator channel. The definition of DA directly implies the need to optimize the dynamics of the "worst particle" in the beam, which leads to a functional of type (4). We also note that transfer matrices or transfer maps can be used to obtain discrete equations of particle dynamics [14].

REFERENCES

[1] A. I. Propoj, Elements of the Theory of Optimal Discrete Processes. Moscow: Nauka, 1973 (In Russian).

[2] D. A. Ovsyannikov, Modeling and Optimization of Charged Particle Beam Dynamics. Leningrad: Leningrad State University Publishing House, 1990 (In Russian).

[3] E. D. Kotina and A. D. Ovsyannikov, "On simultaneous optimization of programmed and perturbed motions in discrete systems", in *Proc. 11th Int. IFAC Workshop on Control Applications of Optimization (CAO 2000)*. St. Petersburg, Russia, July 2000, pp. 183-185.

[4] E. D. Kotina, "Control discrete systems and their applications to beam dynamics optimization", in *Proc. 1st Int. Conf. Physics and Control (PhysCon 2003)*. St. Petersburg, Russia, August 2003, pp. 997–1002, 12370411

[5] E. D. Kotina, "Discrete optimization problem in beam dynamics", *Nucl. Instrum. Methods*, vol. 558, no. 1, pp. 292-294, 2006.

[6] A. D. Ovsyannikov, "Control of program and disturbed motions", *Vestnik of Saint Petersburg University. Series 10. Applied Mathematics. Computer Science. Control Processes*, no. 4, pp. 111-124, 2006.

[7] D. A. Ovsyannikov *et. al.*, "Beam dynamics optimization: models, methods and applications", *Nucl. Instrum. Methods,* vol. 558, no. 1, pp. 11-19, 2006.

[8] M. Yu. Balabanov, M. A. Mizintseva and D. A. Ovsyannikov, "Beam dynamics optimization in a linear accelerator", *Vestnik of saint Petersburg University. Applied Mathematics. Computer Science. Control Processes*, vol. 14, no. 1, pp. 4-13, 2018.

[9] D. A. Ovsyannikov *et. al.*, "Optimization of dynamics of trajectory bundles using smooth and nonsmooth functionals. Part 1", *Vestnik of Saint Petersburg State University. Applied Mathematics. Computer Science. Control Processes.* vol. 16, no. 1, pp. 73-84, 2020.
doi.org /10.21638/11702/spbu10.2020.107

[10] E. D. Kotina and D. A. Ovsyannikov, "Mathematical model of joint optimization of programmed and perturbed motions in discrete systems", *Vestnik of Saint Petersburg State University. Applied Mathematics. Computer Science. Control Processes.* vol. 17, no. 2, pp. 213-224, 2021.
doi.org /10.21638/11701/spbu10.2021.210

[11] A. A. Poklonskiy, D. NeuJJer, M. Berz, D. A. Ovsyannikov and A. D. Ovsyannikov, "Optimizing buncher and phase rotator for Neutrino Factory", *Vestnik of Saint Petersburg University. Series 10. Applied Mathematics. Computer Science. Control Processes*, no. 1, pp. 64-76, 2006.

[12] P. V. Snopok, D. A. Ovsyannikov, A. D. Ovsyannikov, C. J. Johnstone and M. Berz, "High-order simulation of muon collider interaction region", *Vestnik of Saint Petersburg University. Series 10. Applied Mathematics. Computer Science. Control Processes*, no. 1, pp. 77-83, 2006.

[13] A. V. Bogomyagkov et. al., "Dynamic aperture of the NICA collider optimized with a genetic algorithm", Physics of Particles and Nuclei Letters, vol. 16, no. 1, pp. 21-29, 2019.
doi: 10.1134/S1547477119010060

[14] M. Berz, Modern Map Methods in Particle Beam Physics. San Diego, USA: Academic Press, 1999.