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ELECTRODYNAMICS OF WEAKLY COUPLED RF CAVITIES

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Abstract

The configuration formed by a pair of coupled identical rf cavities may manifest unexpected electromagnetic properties concerning its interaction with external rf sources and charged particle beams. Coupling splits resonance frequency of such a system into two different frequencies corresponding to different modes – 0 and π. If cavity walls losses take place, that is cavities Q-values are finite for both modes, composite cavity resonance curves may overlap. This means that both cavity modes are excited effectively by external rf generator or by an intensity modulated charged particles beam traversing cavities. Thus, mixed mode excitation takes place, and resonance properties of such mixed mode as well as beam current loading effect differ qualitatively and quantitatively from those for real cavity eigen modes.

In this paper, field approach is used to describe two coupled cavities system behavior. The expression for any cavity mode amplitude allows obtaining frequency dependence of cavity mixed mode amplitude for both cavities of the system under consideration. Appropriate formulae are supplemented by plots and vector diagrams.

INTRODUCTION

Let us consider rf circuit consisting of two coupled identical rf cavities Here and in following analysis only the main TM mode of each cavity having the lowest eigen frequency is taken into account (TM_{010} in the case of pill box cavities). This unit has two resonance frequencies corresponding to the different composite cavity modes – 0 and π. In the first case, the field phases in both cavities are the same, while in other one components of rf fields are opposite in neighboring cells. The following frequency dependence for field amplitude A_i and field phase φ_i takes phase for this composite cavity excited by external rf generator

$$A_i(\omega) = \frac{a_i}{\sqrt{1+\left(2Q\dfrac{\omega-\omega_i}{\omega_i}\right)^2}};$$

$$\tan\varphi_i(\omega) = 2Q\frac{\omega-\omega_i}{\omega_i} \quad i=1,2 \qquad (1)$$

Q_i and ω being quality factor and frequency respectively, while a_i stands for field amplitude at resonance for each mode.

Let us assume that Q-value as well as a_i are the same for both modes and the cavity is excited at the frequency ω_r satisfying the conditions

$$\tan\varphi_i(\omega) = 2Q\frac{\omega_r-\omega_i}{\omega_i} = \pm 1 \qquad (2)$$

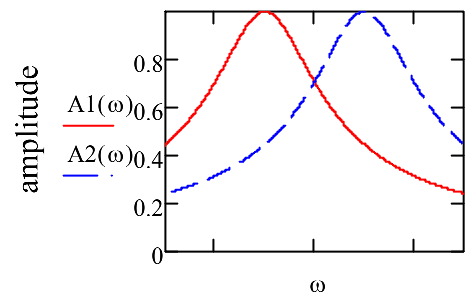

Figure 1: Zero and π modes two coupled cavities.

In this particular case resonance curves for both modes overlap as it is shown on Fig. 1. Appropriate vector diagram on complex plane is shown on Fig. 2 for both coupled cavities. One might say that united cavity operates at $\pi/2$ mode since as it is seen from diagram the appropriate phase shift between both cavities is equal to this value but is not the case. Resonance conditions take place at the frequency ω_r, but frequency mentioned is not the cavity eigen frequency.

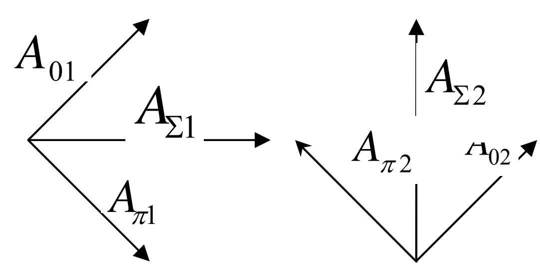

Figure 2: Vector diagrams of voltages in weakly coupled cavities.

Assuming the cavity 1 being excited by an external rf generator over feeder or by modulated charged particles beam let us consider charged beam loading properties of both cavities, beam being accelerated in the cavity 2. We consider that pill box cavities are coupled over an aperture in cylinder part of cavities walls (so called parallel coupling). Induced by accelerated beam rf voltage (or any field component) in cavity 2 is determined by first term of equation (1), while $\varphi_i = \dfrac{\pi}{4}+\pi,\, i=1,2$ (We consider that charged particles are accelerated on the crests of rf voltage excited in cavity 2 by external rf

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generator) In example considered the voltage induced in cavity 2 by external rf generator is opposite to the second one induced by the beam. This in turn means that according vector diagram the total voltage amplitude in cavity 1 is the sum of amplitudes of voltages mentioned. Thus, while beam loading results in accelerating voltage reduction in accelerating cavity appropriate field growth takes place in feeding rf cavity.

RF FIELD INDUCED BY MOOVING SINGLE CHARGED BUNCH

Following the rf cavity excitation theory [1] we shall derive an expression for the voltage that charged bunch leaves in cavity after its pass. According to the theory mentioned vector potential $\vec{A}(\vec{r},t)$ of induced field is the expansion on cavity eigen functions $\vec{A}_\lambda(\vec{r})$ with time dependent coefficients $g_\lambda(t)$:

$$\vec{A}(\vec{r},t)=\sum_{\lambda=1}^{\infty} g_\lambda(t)\vec{A}_\lambda(\vec{r}) \qquad (3)$$

satisfying the equations

$$\Delta\vec{A}_\lambda(\vec{r})+k_\lambda{}^2\vec{A}_\lambda(\vec{r})=0 \qquad (4)$$

$$\frac{d^2g_\lambda(t)}{dt^2}+\frac{\omega_\lambda}{Q_\lambda}\frac{dg_\lambda}{dt}+\omega_\lambda{}^2 g_\lambda(t)=\int_V \vec{j}(\vec{r}.t)\vec{A}_\lambda(\vec{r})dV \qquad (5)$$

and the boundary conditions $(\vec{A}_\lambda\vec{n})=0$ on cavity surface, $\vec{n}$ and $\vec{j}(\vec{r},t)$ being vector normal to cavity surface and bunch current density respectively, $k_\lambda=\omega_\lambda/c$, c - light velocity, ω_λ and Q_λ - resonance frequencies of individual modes and their quality factors. Integration is performed over cavity volume V. It is assumed that eigen functions are normalized by the condition

$$\int_V A_\lambda^2=\mu_0c^2=1/\varepsilon_0 \qquad (6)$$

For the single charge q having zero dimensions and constant velocity v inside the cavity specific solution for $g(t)$, satisfying initial conditions $g(0)=\dot{g}(0)=0$ (corresponding equal to zero electric and magnetic components of induced field) can be represented in the form [2]:

$$g(t)=\frac{1}{\omega}\int_0^{L/v} J(\tau)\sin\omega(t-\tau)d\tau= \frac{\sin\omega t}{\omega}J_1-\frac{\cos\omega t}{\omega}J_2, \qquad (7)$$

where

$$J_1=\int_0^{L/v} J(\tau)\cos\omega\tau\, d\tau,\quad J_2=\int_0^{L/v} J(\tau)\sin\omega\tau\, d\tau \qquad (8)$$

$$J(t)=qvA(vt)\eta(t)\eta(L-vt), \qquad (9)$$

$\eta()$ being Heaviside step function.

This specific solution is valid for time interval $t>L/v$ and zero cavity losses as well.

Let us assume that a particle carrying a charge e enters the cavity at moment $t=0$ and $g(t)=a\sin(\omega t+\varphi)$. Then the energy E_{ac} acquired by this probe charge after traversing the cavity may be represented as

$$\mathrm{E}_{ac}(\varphi)=-e\int \dot{g}(t)A(z)dz= -\frac{ea\omega}{q}\cos\varphi J_1+\frac{ea\omega}{q}\sin\varphi J_2 \qquad (10)$$

Representing rf cavity in the form of equivalent thin gap of zero length (accelerating gap) with applied rf voltage one can conclude that appropriate voltage amplitude U_m is equal to

$$U_m=\frac{\mathrm{E}_{ac}}{e}=\frac{a\omega}{q}\left(J_1^2+J_2^2\right)^{1/2} \qquad (11)$$

This can be expressed in terms of cavity shunt impedance R and cavity quality factor Q:

$$R=\frac{U_m^2}{P_0},\ Q_0=\frac{\omega W}{P_0} \qquad (12)$$

where P_0 stands for cavity walls power losses and W is electromagnetic energy stored in the cavity volume.

$$W=\frac{\varepsilon_0}{2}\int_V E_m^2dV=\frac{a^2\omega^2\varepsilon_0}{2}\int_V A^2(\vec{r})dV \qquad (13)$$

Taking into account normalization condition one arrives finally at relations

$$W=\frac{a^2\omega^2}{2},\ J_1^2+J_2^2=\frac{R}{Q_0}\frac{\omega q^2}{2} \qquad (14)$$

Let us calculate energy loss for the particle traversing cavity filled with the field induced by previous charge, both radiating charge and probe particle being spaced by time interval equal to period of rf oscillations.

$$\mathrm{E}_{lost}=-ev\int_0^{L/v}\dot{g}(t)A(vt)dt= -ev\int_0^{L/v}\left[\int_0^{L/v}J(\tau)\cos(\omega t-\omega\tau)d\tau\right]A(vt)dt =-\frac{e}{q}\left(J_1^2+J_2^2\right) \qquad (15)$$

Together with last expression this gives

$$\mathrm{E}_{lost}=-\frac{eq\omega}{2}\frac{R}{Q_0} \qquad (16)$$

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In terms of thin gap this means that bunch with charge q induces rf voltage of amplitude

$$U = \frac{q\omega}{2}\frac{R}{Q_0} \quad (17)$$

and rf phase π. Furthermore, taking into account field damping we arrive finally to the next expression for rf field, induced by charged bunch on equivalent thin gap

$$U = -\frac{q\omega}{2}\frac{R}{Q_0}\exp(-\omega t/2Q)\cos\omega t \quad (18)$$

Very often, current value I averaged over RF period is used instead of charge value

$$U = -\pi I\frac{R}{Q_0}\exp(-\omega t/2Q)\cos\omega t \quad (19)$$

MIXED MODE OF WEAKLY COUPLED RF CAVITIES

We proceed now to qualitative analysis of the rf circuit consisting of two parallel coupled rf cavities. The cavity 1 is excited by infinite sequence of moving charged particles bunches, ω being appropriate passing frequency. Starting from last but one expression for the rf voltage induced on equivalent thin gap by single charge and summing all voltages induced by different bunches one has for steady state for any of two ($0\, and\, \pi$) excited main accelerated modes in cavity 1

$$U_{1i}(t) = -\frac{IRQ_i}{Q_{0i}}\frac{\cos(\omega t+\varphi_i)}{\sqrt{1+\left(2Q_i\frac{\omega-\omega_i}{\omega_i}\right)^2}} \quad (20)$$

$$\tan(\varphi_i) = 2Q_i\frac{\omega-\omega_i}{\omega_i},\ i1,2 \quad (21)$$

where Q_0 and Q stand for unloaded and loaded quality factors. The notation U_{1i} means rf voltage induced by infinite sequence of cared bunches in cavity 1 for zero ($i=1$) and $\pi(i=2)$ mode respectively.

It is convenient to use new set of variables and parameters

$$\omega_1+\omega_2 = 2\omega_0,\ \omega_2-\omega_1 = 2\Delta$$
$$\frac{\omega-\omega_0}{\omega_0} = \frac{2\varepsilon}{Q_1+Q_2},\ \Delta/\omega_0 = \frac{2\delta}{Q_1+Q_2} \quad (22)$$

In these notations the amplitudes and phases of two excited modes in cavity 1 are

$$A_1 = \frac{U_1}{\sqrt{1+\left[\frac{4Q_1}{Q_1+Q_2}(\varepsilon+\delta)\right]^2}} \quad (23)$$

$$A_2 = \frac{U_2}{\sqrt{1+\left[\frac{4Q_2}{Q_1+Q_2}(\varepsilon-\delta)\right]^2}} \quad (24)$$

$$\tan\varphi_1 = \frac{4Q_1}{Q_1+Q_2}(\varepsilon+\delta) \quad (24)$$

$$\tan\varphi_2 = \frac{4Q_2}{Q_1+Q_2}(\varepsilon-\delta) \quad (25)$$

$$U_1 = I\rho_1 Q_1,\ \rho_1 = R_1/Q_{01} \quad (26)$$

$$U_2 = I\rho_2 Q_2,\ \rho_{21} = R_2/Q_{02} \quad (27)$$

The amplitude and phase of the total rf field in cavity 1

$$A_\Sigma^2(\varepsilon) = A_1^2(\varepsilon)+A_2^2(\varepsilon)+ 2A_1(\varepsilon)A_2(\varepsilon)\cos[\varphi_1(\varepsilon)-\varphi_2(\varepsilon)] \quad (28)$$

$$\tan[\varphi_\Sigma(\varepsilon)] = \frac{A_1(\varepsilon)\sin[\varphi_1(\varepsilon)]+A_2(\varepsilon)\sin[\varphi_2(\varepsilon)]}{A_1(\varepsilon)\cos[\varphi_1(\varepsilon)]+A_2(\varepsilon)\cos[\varphi_2(\varepsilon)]} \quad (29)$$

The same expressions are used to obtain amplitude and phase in cavity 2, the phase $\varphi_2+\pi$ has to be substituted in upper formulae instead of φ_2. Fig.3 Illustrates frequency dependence of amplitudes in both cavities and the phase shift between two voltages as well

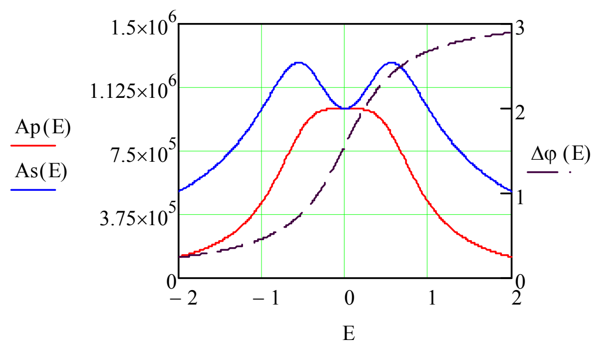

Figure 3: Frequency dependence of amplitudes in cavity 1 (As), cavity 2 (Ap) and phase shift between cavities voltages.

CONCLUSION

Weakly coupled rf cavities operate in so called mixed mode. Frequency as well beam loading properties of this mode differ from real modes

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