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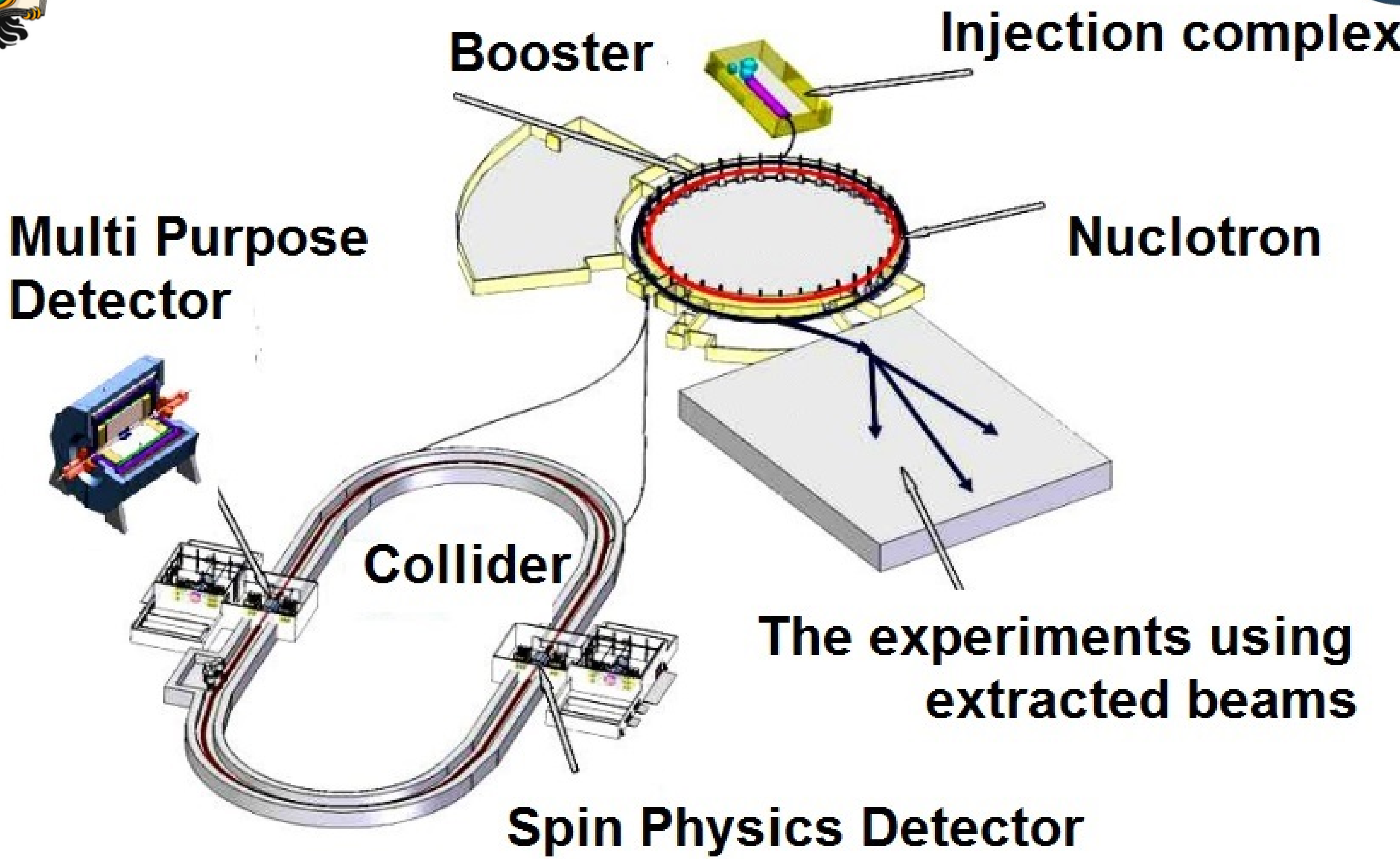
# Some Problems of the Beam Extraction from Circular Accelerators

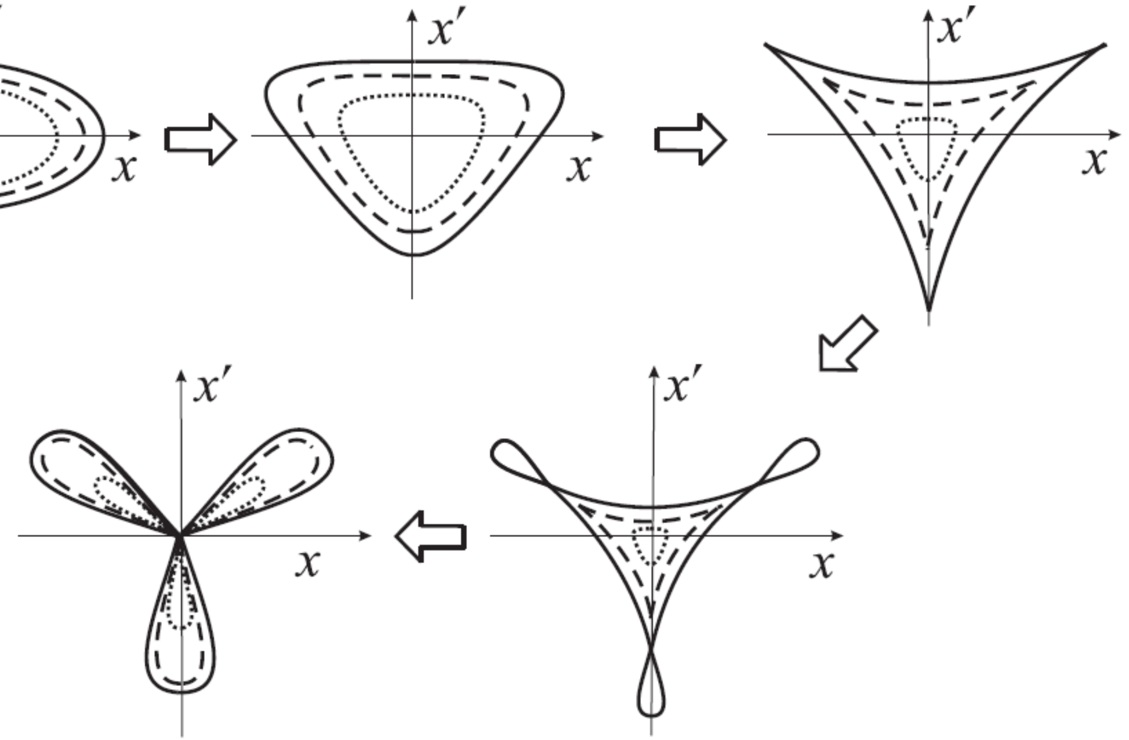

Booster
Injection complex
Multi Purpose
Detector
Nuclotron
Collider
The experiments using
extracted beams
Spin Physics Detector

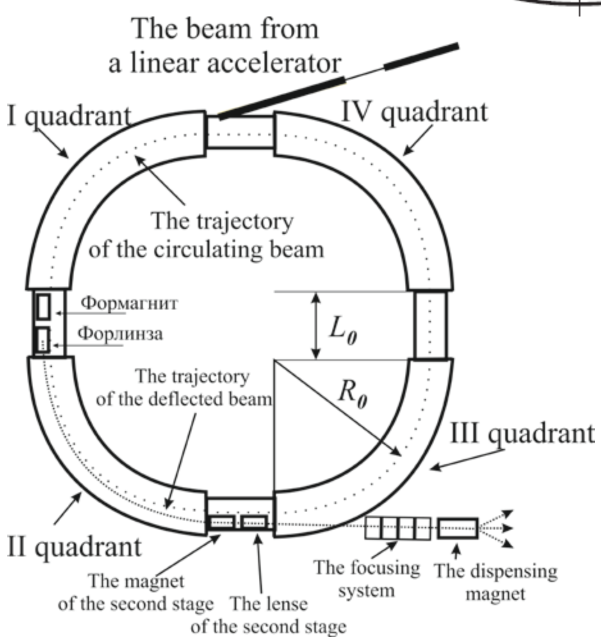
# From physical model to mathematical models

1. **The accuracy of approximation of the ideal mapping** generated by the dynamical system: how estimate the closeness of ideal solutions and the corresponding approximate solutions?
2. **Preserving the qualitative properties** inherent in the dynamical system under study:
   – symplectic property for Hamiltonian systems;
   – conservation of exact and approximate integrals of motion and so on.
3. **Constructing accurate maps** for some practical classes of dynamical systems.
4. **The dynamics of the beam** as an ensemble.
5. The problem of **parallel and distributed computing**.

# Physical models

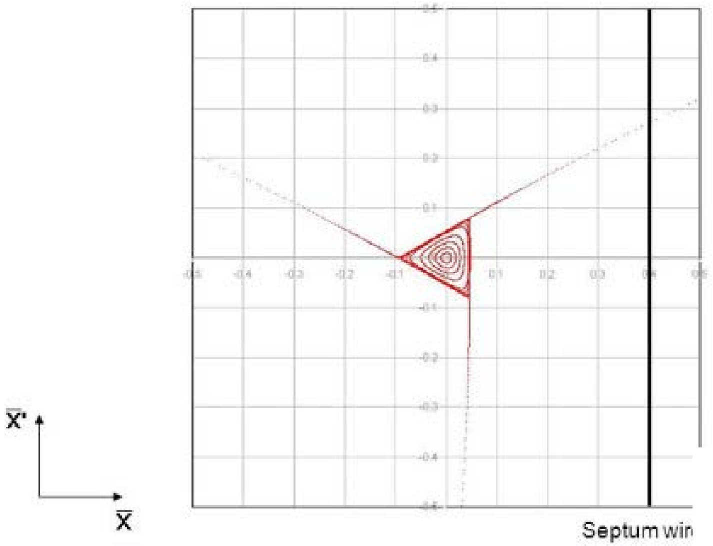
## Slow extraction using third-order resonance

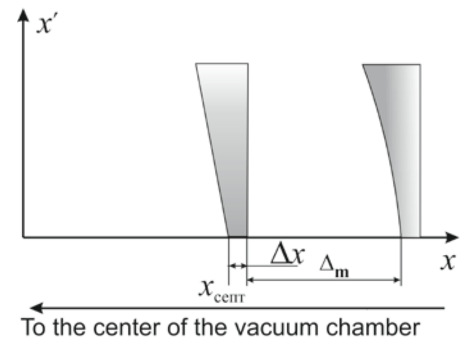


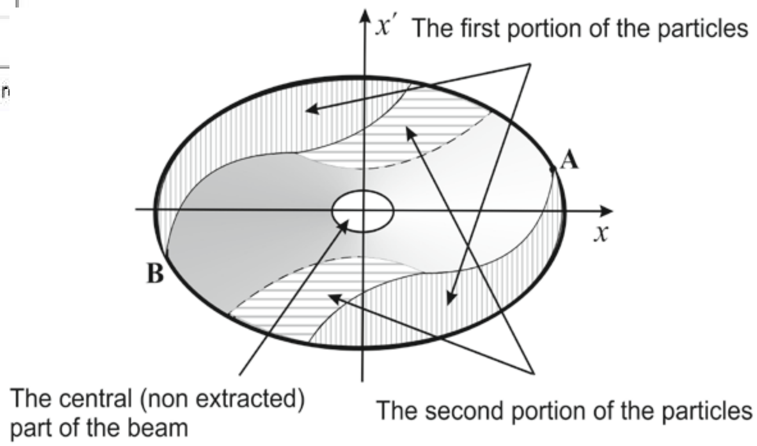



# Physical models

## Slow extraction using third-order resonance

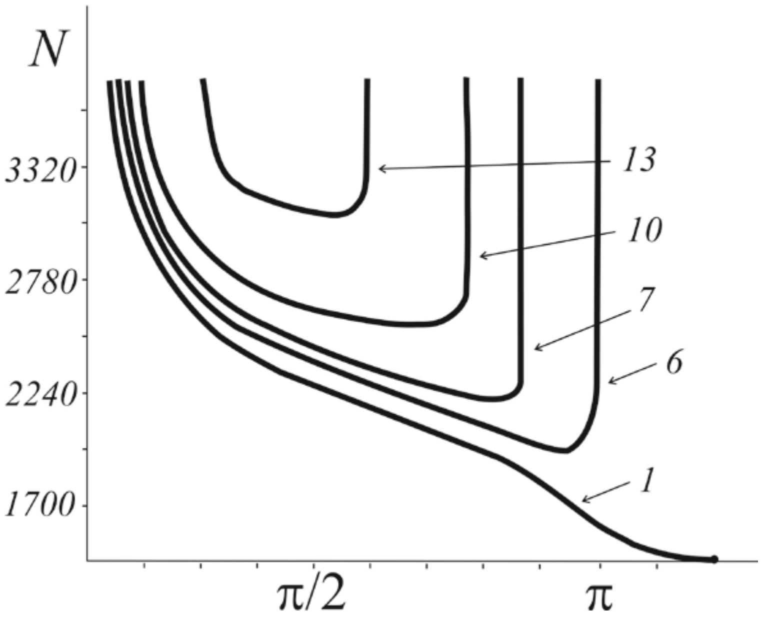


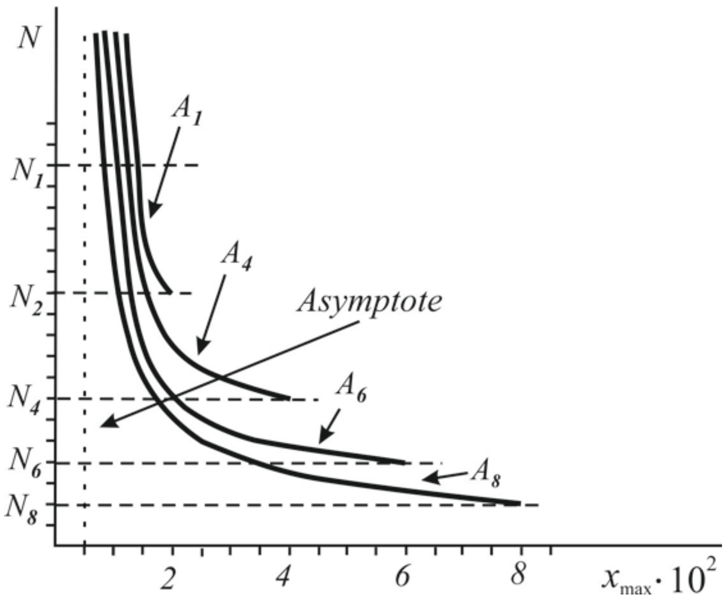



# Physical models

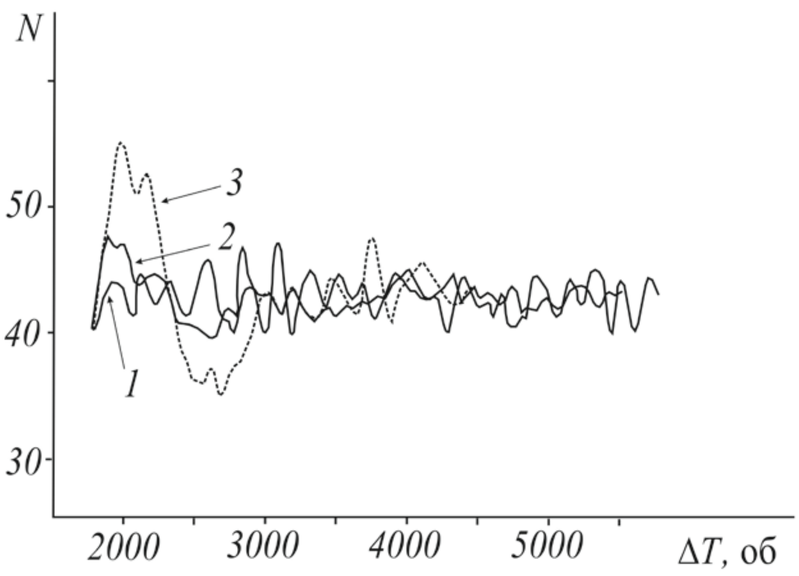
## Slow extraction using third-order resonance

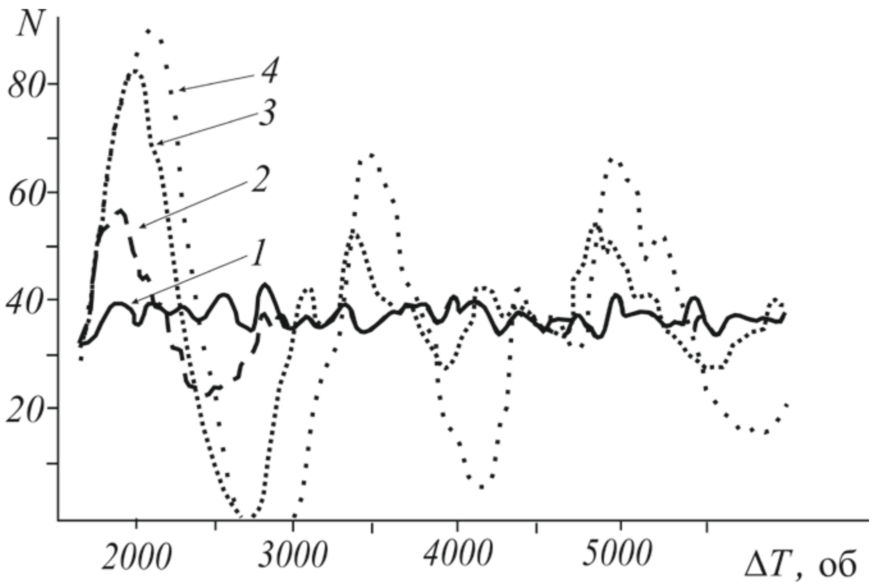


# Physical models

## Slow extraction using third-order resonance




**Inertial properties of the beam for different amplitudes of the feedback impulse**

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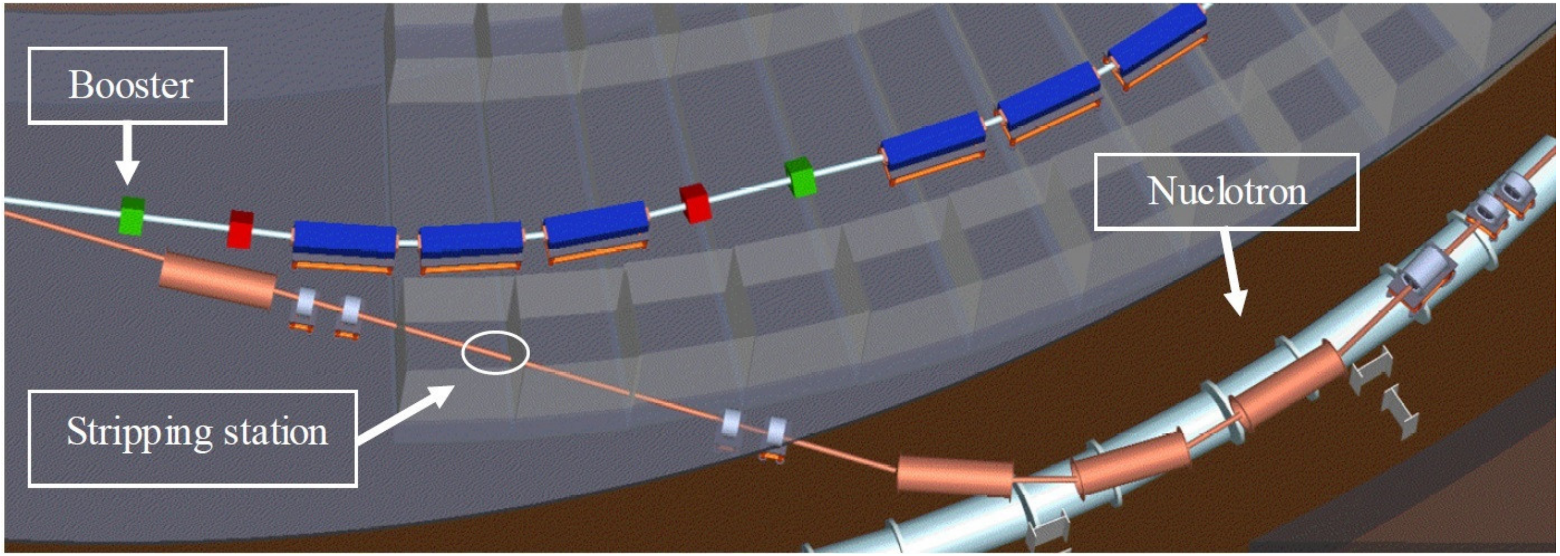
# Physical models

## Fast extraction from Booster to Nuclotron

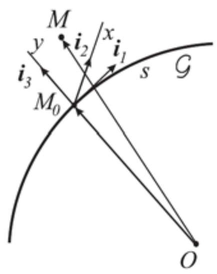



The transport channel Booster-into-Nuclotron has a three-dimensional geometry: magnetic elements do not lie in the same plane. In this case we should use the total motion equation:

- the increasing of emittance is a result of the **parameters mismatch**;
- the increasing of emittance is a result of the **error dispersion**;
- the type of particles - **ions of gold with charge**.

# Mathematical models -1

## Evolution equations (coordinates)




$$\mathbf{v} = \frac{d\mathbf{R}}{dt}, \qquad \frac{d(m\mathbf{v})}{dt} = q\left(\mathbf{E} + [\mathbf{v} \times \mathbf{B}]\right),$$

$$B_x = \frac{\partial \psi}{\partial x}, \quad B_y = \frac{\partial \psi}{\partial y}, \quad B_s = \frac{1}{1+hx}\frac{\partial \psi}{\partial s},$$

$$\psi(x,y,s) = \sum_{k,i=0}^{\infty} a_{ik}(s)\frac{x^i}{i!}\frac{y^k}{k!}.$$

$$x'' - h^2 x = -h - \frac{x'(2hx'+h'x)}{1-hx} - \frac{1}{c\beta\gamma(1-hx)}\left((1-hx)^2 + x'^2 + y'^2\right)^{1/2} \times$$
$$\times \left(x'y'B_x - \left((1-hx)^2 + x'^2\right)B_y + (1-hx)y'B_s\right),$$

$$y'' = -\frac{y'(2hx'+h'x)}{1-hx} + \frac{1}{c\beta\gamma(1-hx)}\left((1-hx)^2 + x'^2 + y'^2\right)^{1/2} \times$$
$$\times \left(-x'y'B_y - \left((1-hx)^2 + y'^2\right)B_x - (1-hx)x'B_s\right).$$

$$a''_{ik} + kha''_{i,k-1} - kh'a'_{i,k-1} + a_{i+2,k} + a_{i,k+2} +$$
$$+ (3k+1)ha_{i,k+1} + k(3k-1)h^2 a_{ik} + k(k-1)^2 h^3 a_{i,k-1} +$$
$$+ 3kha_{i+2,k-1} + 3k(k-1)h^2 a_{i+2,k-2} + k(k-1)(k-2)h^3 a_{i+2,k-3} = 0.$$

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X},t) = \mathbf{F}^{\mathrm{ext}}(\mathbf{B}^{\mathrm{ext}}(\mathbf{X},t), \mathbf{E}^{\mathrm{ext}}(\mathbf{X},t), \mathbf{X}, t) + \mathbf{F}^{\mathrm{self}}(\langle f(\mathbf{X},t)\rangle_{\mathfrak{M}}, \mathbf{X}, t)$$

# Mathematical models -2

## Matrix Formalism - motion equations

$$d\mathbf{X}/dt = \mathbf{F}(\mathbf{X},t),\ \mathbf{F}(0,t) \equiv 0 \longrightarrow \frac{d\mathbf{X}}{dt} = \sum_{k=1}^{\infty} \mathbb{P}^{1k}\mathbf{X}^{[k]} = \sum_{k=0}^{\infty} \frac{\partial^k \mathbf{F}(0,t)}{\partial \mathbf{X}^k}\frac{\mathbf{X}^k}{(k)!}$$

*Kronecker product*

$$\mathbf{X}^{[k]} = \underbrace{\mathbf{X}\otimes\ldots\otimes\mathbf{X}}_{k\text{-times}}.$$

$$\frac{d\mathbf{X}^{[k]}}{dt} = \sum_{k=0}^{k} \mathbf{X}^{[j]} \otimes \frac{d\mathbf{X}}{dt} \otimes \mathbf{X}^{[k-j-1]} = \sum_{k=1}^{\infty}\sum_{j=0}^{k} \mathbf{X}^{[j]} \otimes \mathbb{P}^{1k} \otimes \mathbf{X}^{[k]}.$$

$$\mathbb{P}^{kj} = \mathbb{P}^{1\,(j-k+1)} \oplus \mathbb{P}^{(k-1)\,(j-1)}, \quad j \geq k,$$

$$\mathbb{P}^{k\,k} = \mathbb{P}^{1\,1} \oplus \mathbb{P}^{(k-1)\,(k-1)} = \left(\mathbb{P}^{11}\right)^{\oplus k}, \quad k \geq 2.$$

$$\frac{d\mathbf{X}^{[k]}}{dt} = \sum_{j=k}^{\infty} \mathbb{P}^{kj}\mathbf{X}^{[j]}$$

$$\frac{d\mathbf{X}^{\infty}}{dt} = \mathbb{P}^{\infty}(t)\mathbf{X}^{\infty} \qquad \mathbb{P}^{\infty} = \begin{pmatrix} \mathbb{P}^{11} & \mathbb{P}^{12} & \ldots & \mathbb{P}^{1k} & \ldots \\ \mathbb{O} & \mathbb{P}^{22} & \ldots & \mathbb{P}^{2k} & \ldots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ \mathbb{O} & \mathbb{O} & \ldots & \mathbb{P}^{2k} & \ldots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

# Mathematical models -3

## Matrix Formalism - reverse evolutionary matrices

$$\mathbf{X}^{\infty} = \mathbb{R}^{\infty}(t)\mathbf{X}_0^{\infty} \quad \Longleftrightarrow \quad \mathbf{X}(t) = \sum_{k=1}^{\infty} \mathbb{R}^{1k}(t|t_0)\mathbf{X}_0^{[k]},$$

$$\mathbb{R}^{ik}(t|t_0) = \sum_{j=i+1}^{k} \int_{t_0}^{t} \mathbb{R}^{ii}(t|\tau)\mathbb{P}^{ij}(\tau)\mathbb{R}^{jk}(\tau|t_0)d\tau, \quad \mathbb{R}^{ii}(t|t_0) = \left(\mathbb{R}^{11}(t|t_0)\right)^{[i]}$$

$$(\mathbb{R}^{\infty})^{-1} = \mathbb{T}^{\infty} = \begin{pmatrix} \mathbb{T}^{11} & \mathbb{T}^{12} & \dots & \mathbb{T}^{1k} & \dots \\ \mathbb{O} & \mathbb{T}^{22} & \dots & \mathbb{T}^{2k} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \mathbb{T}^{kk} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \quad \mathbb{T}^{kk} = \left(\mathbb{R}^{kk}\right)^{-1} = \left(\mathbb{R}^{11}\right)^{-[k]} = \left(\left(\mathbb{R}^{11}\right)^{-1}\right)^{[k]},$$

$$\mathbb{T}^{ik} = -\sum_{l=i}^{k-i} \mathbb{T}^{il}\mathbb{R}^{lk}\mathbb{T}^{kk}, \quad i < k.$$

# Mathematical models -3

## Matrix Formalism – for envelope nonlinear matrices

$$\mathfrak{S}^{ik}(t) = \int\limits_{\mathfrak{M}(t)} f(\mathbf{X},t)\mathbf{X}^{[i]}\left(\mathbf{X}^{[k]}\right)^{*} d\mathbf{X}$$

Here $f(\mathbf{X},t)$ the distribution function on the set $\mathfrak{M}(t)$, occupied by particles.

$$\frac{d\mathfrak{S}^{ik}}{dt} = \sum_{l=i}^{\infty}\mathbb{P}^{il}\mathfrak{S}^{lk} + \sum_{l=k}^{\infty}\mathfrak{S}^{lk}\left(\mathbb{P}^{kl}\right)^{*}, \quad i,k \le 1 \qquad \frac{d\mathbf{X}^{[k]}}{dt} = \sum_{j=k}^{\infty}\mathbb{P}^{kj}\mathbf{X}^{[j]}$$

$$\mathfrak{S}^{ik}(t) = \sum_{l=i}^{\infty}\sum_{j=k}^{\infty}\mathbb{R}^{il}(t|t_0)\mathfrak{S}_0^{lj}\left(\mathbb{R}^{kj}(t|t_0)\right)^{*}$$

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# Mathematical models -3
# The matrix presentation using Lie algebraic tools

According to the well known Lie algebraic tools[1] the our motion equations can be written using so called Lie map (an evolution operator in the exponential form)

$$\frac{d\mathcal{M}(t|t_0)}{dt} = \mathcal{V}(t) \circ \mathcal{M}(t|t_0),$$ with the initial condition $\mathcal{M}(t_0|t_0) = \mathcal{I}d \;\; \forall\, t_0 \in \mathfrak{T}$

where $\mathcal{V}(t) = \mathcal{L}_{\mathbf{F}} = \mathbf{F}^*(\mathbf{X})\frac{\partial}{\partial \mathbf{X}} = \sum_{k=0}^{\infty}\left(\mathbf{X}^{[k]}\right)^*(\mathbf{F}_k)^*\frac{\partial}{\partial \mathbf{X}} = \sum_{k=0}^{\infty}\mathcal{L}_{\mathbf{F}}^k$ is a Lie operator.

The solution the operator equation can be written in the form of chronological ordered series (Volterra series)

$$\mathcal{M}(t|t_0) = \mathcal{I}d + \sum_{k=1}^{\infty}\int_{t_0}^{t}\int_{t_0}^{\tau_1}\ldots\int_{t_0}^{\tau_{k-1}}\mathcal{V}(\tau_k)\circ\mathcal{V}(\tau_{k-1})\circ\ldots\circ\mathcal{V}(\tau_1)\,d\tau_k\ldots d\tau_1$$

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[1] **See, for example, A.J.Dragt *Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics*.** University of Maryland, College Park. www.physics.umd.edu/dsat/.

# Mathematical models -3
# Hamiltonian formalism

In the case of Hamiltonian motion equation we can write $\dfrac{d\mathbf{X}}{dt} = \mathbb{J}(\mathbf{X})\dfrac{\partial\mathcal{H}(\mathbf{X},t)}{\partial\mathbf{X}}$, where

$$\mathcal{H} = \sum_{k=2}^{\infty}\mathbf{H}_k^*(t)\mathbf{X}^{[k]} = -(1+hx)\frac{q}{\mathcal{E}_0}A_s -$$

$$-(1+hx)\left[(1+\eta)^2 - \left(\frac{m_0c^2}{\mathcal{E}_0}\right)^2 - \left(P_x - \frac{q}{c}A_x\right)^2 - \left(P_y - \frac{q}{c}A_y\right)^2\right]^{1/2}$$

One can write $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \sum_{k=3}^{\infty}\varepsilon^{k-2}\mathcal{H}_k,$ where $\mathcal{H}_k$ are homogeneous polynomials of *k*-th order. Here $\mathbf{H}_k(t)$ vectors of coefficients for these polynomials.

This decomposition leads to expansion for motion equation of the corresponding series in according Dragt's approach. After this one can write

$$\mathcal{M}\left(t|t_0;\sum_{k=1}^{\infty}\mathcal{L}_{\mathbf{F}_k}\right) = \ldots\circ\mathcal{M}\left(t|t_0;\mathcal{L}_{\mathbf{G}_k}\right)\circ\mathcal{M}\left(t|t_0;\mathcal{L}_{\mathbf{G}_{k-1}}\right)\circ\ldots$$

$$\ldots\circ\mathcal{M}\left(t|t_0;\mathcal{L}_{\mathbf{G}_2}\right)\circ\mathcal{M}\left(t|t_0;\mathcal{L}_{\mathbf{G}_1}\right),$$

# Mathematical models -3
# Magnus presentation

The chronological series is not convenient for practical computation. Instead of this series there is used so called Magnus presentation for Lie map

$$\mathcal{M}(t|t_0) = \exp \mathcal{W}(t|t_0; \mathcal{V})$$

$$\mathcal{W}(t|t_0) = \int_{t_0}^{t} \mathcal{V}(\tau)d\tau + \alpha_1 \int_{t_0}^{t} \left\{ \mathcal{V}(\tau), \int_{t_0}^{\tau} \mathcal{V}(\tau') \right\} d\tau +$$

$$+ \alpha_1^2 \int_{t_0}^{t} \left\{ \mathcal{V}(\tau), \int_{t_0}^{\tau} \left\{ \mathcal{V}(\tau'), \int_{t_0}^{\tau'} \mathcal{V}(\tau'')d\tau'' \right\} d\tau' \right\} d\tau +$$

$$+ \alpha_1 \alpha_2 \int_{t_0}^{t} \left\{ \left\{ \mathcal{V}(\tau), \int_{t_0}^{\tau} \mathcal{V}(\tau')d\tau' \right\}, \int_{t_0}^{\tau} \mathcal{V}(\tau')d\tau' \right\} d\tau + \dots$$

Here is a commutator for any two operators. Similar formulae can be evaluated up to any order.

# Mathematical models -3
# Magnus presentation

One can introduce the following presentation for a new operator

$$\mathcal{W}_\lambda(t|t_0;\mathcal{V}) = \sum_{k=1}^{\infty} \lambda^k \mathcal{W}_k(t|t_0;\mathcal{V})$$

After some transformation we can obtain the following family of equalities:

$$\mathcal{W}_1(t|t_0;\mathcal{V}) = \int_{t_0}^{t} \mathcal{V}(\tau)d\tau,$$

$$\mathcal{W}_2(t|t_0;\mathcal{V}) = -\frac{1}{2}\int_{t_0}^{t}\int_{t_0}^{\tau} \{\mathcal{V}(\tau),\mathcal{V}(\tau')\}\, d\tau' d\tau.$$

$$\mathcal{W}_3(t|t_0;\mathcal{V}) = \frac{1}{6}\int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{\tau'} (\{\{\mathcal{V}(\tau),\mathcal{V}(\tau')\},\mathcal{V}(\tau'')\} +$$

$$+ \{\{\mathcal{V}(\tau''),\mathcal{V}(\tau')\},\mathcal{V}(\tau)\})\, d\tau'' d\tau' d\tau.$$ and so on.

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# Mathematical models -3
# Magnus presentation

We can note the (according to the Ado lemma) every finite-dimensional algebra has faithful finite-dimensional representation. This allows us to use matrix algebras Lie. Using the above mentioned presentations we can obtain the following operator estimation for previous series. For example,

$$\mathcal{W}(t|t_0) = \int_{t_0}^{t} \mathcal{A}(\tau)d\tau + \alpha_1 \int_{t_0}^{t} \left\{ \mathcal{A}(\tau), \int_{t_0}^{\tau} \mathcal{A}(\tau')d\tau' \right\} d\tau + \alpha_1^2 \int_{t_0}^{t} \left\{ \mathcal{A}(\tau), \int_{t_0}^{\tau} \left\{ \mathcal{A}(\tau'), \int_{t_0}^{\tau'} \mathcal{A}(\tau'')d\tau'' \right\} d\tau' \right\} d\tau +$$

$$+ \alpha_1\alpha_2 \int_{t_0}^{t} \left\{ \left\{ \mathcal{A}(\tau), \int_{t_0}^{\tau} \mathcal{A}(\tau')d\tau' \right\}, \int_{t_0}^{\tau} \mathcal{A}(\tau')d\tau'' \right\} d\tau + \ldots$$

Whence it follows

$$\|\mathcal{W}(t|t_0)\| \leq A(t) \left( 1 + 2|\alpha_1|A(t) + 4A^2(t)C_2 + 8A^3(t)C_3 + \sum_{l \geq 4} (2A(t))^l C_l \right),$$

where

$$A(t) = \int_{t_0}^{t} \|\mathcal{A}(\tau)\| d\tau$$

# Mathematical models -3
# Magnus presentation

Here $C_{2l} = \alpha_{2l} + C_{2l-2}C_{2l-4},\ C_{2l+1} = \alpha_{2l} + C_3 C_{2l-1},\ l \geq 2,\ C_0 = 1,\ C_1 = 1,\ C_2 = \alpha_1^2 + |\alpha_1|,$ $C_3 = \alpha_1^3 + 2|\alpha_1|.$ Let be $\mathcal{W} = \sum_{k>0} \mathcal{W}^k,$ where $\mathcal{W}^k$ enclose all *k* nested Lie brackets. Then we

Have the following inequality $\|\mathcal{W}^k(t|t_0)\| \leq A(t)\,(2A(t))^k\, C_k, \quad k \geq 0,$ and for coefficients $\alpha_{2k}$ we have

$$|\alpha_{2m}| \leq \frac{2}{(2\pi)^{2m}} \sum_{k\geq 1} \frac{1}{2^{2k}} < \frac{4}{(2\pi)^{2m}}.$$

Let be $M = \int_{t_0}^{T_2} A(\tau)d\tau,$ then $\|\mathcal{W}^k\|_{L_1} \leq 2^k M^{k+1} C^k$ for all sufficiently great *k.* The majorant series with general members $2^k M^{k+1} C_k$ will be converge (according to D'Alember criterion) if there hold the following inequality

$$\lim_{k\to\infty} \frac{2^{k+1} M^{k+2} C_{k+1}}{2^k M^{k+1} C_k} = q < 1.$$

For $k = 2l,$

$$q = 2M \lim_{l\to\infty} \frac{C_{2l+1}}{C_{2l}} = 2M \lim_{l\to\infty} \frac{\alpha_{2l} + C_3 C_{2l-1}}{\alpha_{2l} + C_{2l-2}C_{2l-4}} = 2M.$$

So the majorizing series will converge on the assumption of $M < 1/2$

**Therefore our series will converge absolutely.**

# Mathematical models -3
# The convergence problem

We can derive corresponding conditions for convergence of matrix formalism for ODE’s. Let cite corresponding estimations.

Let be $\frac{d\mathbf{X}}{dt} = \sum_{k=1}^{\infty} \mathbb{P}^{1k}\mathbf{X}^{[k]}$, from where $\mathbf{X}(t) = \sum_{k=1}^{\infty} \mathbb{R}^{1k}(t|t_0)\mathbf{X}_0^{[k]}$ and we have $\|\mathbf{X}_0\| \leq r$,

and $\|\frac{\partial^k \mathbf{F}(\mathbf{X},\mathbf{U},t)}{\partial x_1^{k_1} \ldots x_n^{k_n}}\| \leq \varphi(t)$, $M = \int_{\mathcal{T}} \varphi(t)dt$, $L = \sup_{t,\tau\in\mathcal{T}} \|\mathbb{R}^{11}(t,\tau)\|$. We can show that

there are the next inequalities $\sup_{t,\tau\in\mathcal{T}} \|\mathbb{R}^{jj}(t,\tau)\| \leq jL^j$ and $\|\mathbb{P}^{ij}(t)\| \leq \frac{\varphi(t)}{(j-1)!}$.

Let be $J_i(L,M) = \begin{cases} \prod_{k=3}^{i} \left\{\frac{L^{k-1}M(k-1)}{(k-2)!} + 1\right\}, i \geq 3, \\ 1, i = 2 \end{cases}$ then we have ( $\overline{\mathbf{X}}$ is an exact solution):

$$\|\overline{\mathbf{X}} - \mathbf{X}_N\| \leq \sum_{k=N+1}^{\infty} \frac{r^k L^{k+1} M k}{(k-1)!} J_k(L,M).$$

# Mathematical models -3
# Preservation of qualitative properties

Usually in beam physics there is used the Hamiltonian formalism for particle beam motion description. This automatically leads us to following equalities $\frac{d\mathbf{X}}{dt} = \mathbb{J}(\mathbf{X})\frac{\partial \mathcal{H}(\mathbf{X},t)}{\partial \mathbf{X}}$, were $\mathbb{J}(\mathbf{X})$ is a symplectic matrix $2n \times 2n$, for example $\mathbb{J}(\mathbf{X}) = \mathbb{J}_0 = \begin{pmatrix} \mathbb{O} & \mathbb{E} \\ -\mathbb{E} & \mathbb{O} \end{pmatrix}$.

The Jacobi matrix $\mathbb{M}(\mathbf{X}, t \,|\, t_0; \mathcal{M}) = \mathbb{M}(\mathbf{X}; t \,|\, t_0) = \frac{\partial \mathcal{M}(t \,|\, t_0; \mathcal{H}) \circ \mathbf{X}}{\partial \mathbf{X}^*}$ satisfies to the following symplecticity condition $\mathbb{M}^*(\mathbf{X}; t \,|\, t_0)\mathbb{J}(\mathbf{X})\mathbb{M}(\mathbf{X}; t \,|\, t_0) = \mathbb{J}(\mathbf{X})$. Here we have $\det \mathbb{M}(\mathbf{X}; t \,|\, t_0) \equiv 1$.
According to the matrix formalism one can derive

$$\mathbb{M}(\mathbf{X}; t \,|\, t_0) = \frac{\partial \mathcal{M}(t \,|\, t_0; \mathcal{H}) \circ \mathbf{X}}{\partial \mathbf{X}^*} = \sum_{k=1}^{\infty} \mathbb{R}^{1k}(t \,|\, t_0; \mathcal{H}) \frac{\partial \mathbf{X}^{[k]}}{\partial \mathbf{X}^*}. \quad (1)$$

Using the Kronecker product and sum properties we can derive

$$\mathbb{M}(\mathbf{X}; t \,|\, t_0) = \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \mathbb{R}^{1k}(t \,|\, t_0; \mathcal{H}) \mathbf{X}^{[j]} \otimes \mathbf{E}_{2n} \otimes \mathbf{X}^{[k-j-1]}. \quad (2)$$

# The Preservation of qualitative properties (qualitative properties)

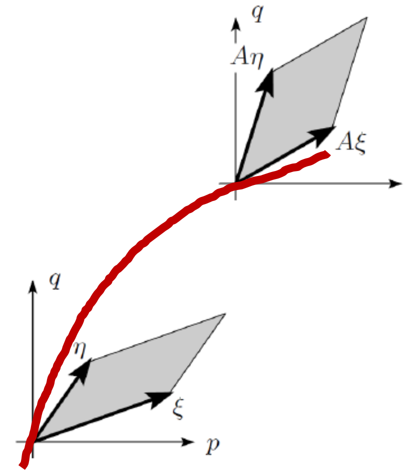
Replacing (2) into (1) one can derive $\mathbb{M}(\mathbf{X};t\,|\,t_0) = \sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\mathbb{R}^{1k}\left(t\,|\,t_0;\mathcal{H}\right)\mathbf{X}^{[j]}\otimes\mathbb{E}\otimes\mathbf{X}^{[k-j-1]}$. From here we can describe

$$\underbrace{\left(\mathbb{R}^{11}\right)^*\mathbb{J}_0\mathbb{R}^{11}}_{k=l=1} + \underbrace{\left(\mathbf{X}\otimes\mathbb{E}+\mathbb{E}\otimes\mathbf{X}\right)^*\left(\mathbb{R}^{12}\right)^*\mathbb{J}_0\mathbb{R}^{11}}_{k=2,\ l=1} +$$

$$+\underbrace{\left(\mathbb{R}^{11}\right)^*\mathbb{J}_0\mathbb{R}^{12}\left(\mathbf{X}\otimes\mathbb{E}+\mathbb{E}\otimes\mathbf{X}\right)}_{k=1,\ l=2} + \sum_{\substack{k,l=1\\k+l>3}}^{\infty}\left(\mathbf{X}^{\odot k}\right)^*\left(\mathbb{R}^{1k}\right)^*\mathbb{J}_0\mathbf{X}^{\odot l} = \mathbb{J}_0,$$

were $\mathbf{X}^{\odot(k-1)} = \sum_{j=0}^{k-1}\mathbf{X}^{[j]}\otimes\mathbb{E}\otimes\mathbf{X}^{[k-j-1]}$. Denoting $\mathbb{R}^{1k} = \mathbb{R}^{11}\mathbb{Q}^{1k}$ (here $\mathbb{Q}^{11} = \mathbb{E}$ ) we derive

$$\sum_{k+l=m}\left(\mathbf{X}^{\odot k}\right)^*\left(\mathbb{Q}^{1\,(k+1)}\right)^*\mathbb{J}_0\mathbb{Q}^{1\,(l+1)}\mathbf{X}^{\odot l} = 0, \quad m\geq 1.$$




Or the following equalities sequence

$$\left(\mathbf{X}^{\odot 1}\right)^*\left(\mathbb{Q}^{12}\right)^*\mathbb{J}_0 + \mathbb{J}_0\mathbb{Q}^{12}\mathbf{X}^{\odot 1} = 0,$$

$$\left(\mathbf{X}^{\odot 2}\right)^*\left(\mathbb{Q}^{13}\right)^*\mathbb{J}_0 + \left(\mathbf{X}^{\odot 1}\right)^*\left(\mathbb{Q}^{12}\right)^*\mathbb{J}_0\mathbb{Q}^{12}\mathbf{X}^{\odot 1} + \mathbb{J}_0\mathbb{Q}^{13}\mathbf{X}^{\odot 2} = 0, \qquad (3)$$

$$\left(\mathbf{X}^{\odot 3}\right)^*\left(\mathbb{Q}^{14}\right)^*\mathbb{J}_0 + \left(\mathbf{X}^{\odot 2}\right)^*\left(\mathbb{Q}^{13}\right)^*\mathbb{J}_0\mathbb{Q}^{12}\mathbf{X}^{\odot 1} +$$
$$+\left(\mathbf{X}^{\odot 1}\right)^*\left(\mathbb{Q}^{12}\right)^*\mathbb{J}_0\mathbb{Q}^{13}\mathbf{X}^{\odot 2} + \mathbb{J}_0\mathbb{Q}^{14}\mathbf{X}^{\odot 3} = 0, \ \ldots$$

# Qualitative properties

The equalities sequences (3) can be solved step-by-step both in analytical and in numerical modes. It should be noted that these equalities impose simple algebraic conditions on corresponding matrix elements. These equalities have the form of linear algebraic equalities with integer coefficients! For example, for four dimensional phase space for the second order matrix $\mathbb{Q}^{[2]} = \{q_{ij}\}$ we obtain

$$\mathbb{Q}^{12} = \begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} & q_{15} & q_{16} & q_{17} & q_{18} & q_{19} & q_{110} \\ q_{21} & q_{22} & q_{23} & q_{15} & q_{25} & q_{17} & q_{27} & q_{19}/2 & 2\,q_{110} & q_{210} \\ q_{31} & q_{32} & q_{33} & -2\,q_{11} & -2\,q_{21} & -q_{12} & -q_{22} & q_{14}/2 & -q_{15} & q_{25}/2 \\ q_{32}/2 & 2\,q_{33} & q_{43} & -q_{12} & -q_{22} & q_{32}/2 & 2\,q_{33} & q_{43} & -q_{12} & -q_{22} \end{bmatrix}.$$

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# Exact solutions -1
## simple example

The correctness of the matrix formalism can be tested for some simple examples.

1. One-dimensional nonlinear equation

$$\frac{dx}{dt} = K_n x^n.$$

The exact solution of this equality has the form

$$x(t) = \sqrt[1-n]{K_n(1-n)(t-t_0) + x_0^{1-n}}.$$

For Lie operator we can write

$$\mathcal{M}(t, t_0) = exp\{(t-t_0)K_n x^n \frac{\partial}{\partial x}\}.$$

After some simple calculations one can obtain the desired expression!

$$\mathcal{M} \circ x_0 = x_0 \sum_{k=0}^{\infty} \binom{\frac{1}{1-n}}{k} ((1-n)K_n(t-t_0)x_0^{n-1})^k = \frac{x_0}{\sqrt[n-1]{1+(1-n)K_n(t-t_0)x_0^{n-1}}}$$

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!

# Exact solutions – 2 correctness of the matrix formalism

The correctness of the matrix formalism can be tested for some simple examples.

2. Fractionally rational solution of nonlinear equations.

In some cases we search the solution of the nonlinear equations in the fractionally rational form

$$\mathcal{M} \circ \mathbf{X}_0 = \mathbf{X}\left(\mathbf{X}_0; t \,|\, t_0\right) = \frac{\mathbf{P}_N\left(\mathbf{X}_0; t \,|\, t_0\right)}{Q_L\left(\mathbf{X}_0; t \,|\, t_0\right)},$$

where

$$\mathbf{P}_N\left(\mathbf{X}_0; t \,|\, t_0\right) = \sum_{k=0}^{N} \mathbb{P}^k(t \,|\, t_0)\mathbf{X}_0^{[k]}, \quad Q_L\left(\mathbf{X}_0; t \,|\, t_0\right) = \sum_{j=0}^{L} \left(\mathbf{Q}_j(t \,|\, t_0)\right)^* \mathbf{X}_0^{[j]}.$$

Let consider the next case $\mathcal{M} = \mathcal{M}_m = \exp\left(t - t_0\right)\mathcal{L}_m$, where $\mathcal{L}_m = \mathbf{G}_m^*(\mathbf{X}_0)\partial/\partial\mathbf{X}_0$.

After some calculation one can obtain

$$\mathcal{M}_m \circ \mathbf{X}_0 = \mathbf{X}_0 + \sum_{k=1}^{\infty} \frac{(t-t_0)^k \mathbb{P}_m^{1k}}{k!} \mathbf{X}_0^{[k(m-1)+1]}, \quad \mathbb{P}_m^{1k} = \prod_{j=1}^{k} \mathbb{G}_m^{\oplus((j-1)(m-1)+1)}.$$

Let us introduce

$$\mathbb{C}_m^k = \left((t-t_0)/(k-1)!\right) \mathbb{P}_m^{1(k-1)}$$

# Exact solutions - 2

then
$$\mathbb{C}^k_m = \mathbb{P}^k_m - \sum_{j=0}^{L} \mathbf{Q}^*_j \otimes \mathbb{C}^{k-j}_m, \quad 1 \le k \le N,$$
$$\mathbb{C}^k_m + \sum_{j=1} \mathbf{Q}^*_j \otimes \mathbb{C}^{k-j}_m = 0, \quad k > N.$$

For the second order nonlinear Hamiltonian equations

$$\frac{dx}{dt} = ax^2, \quad \frac{dP_x}{dt} = bx^2 - 2axP_x, \qquad (4)$$

we can obtain

$$\mathbf{X} = \frac{\mathbf{X}_0 + \mathbb{P}^2_2 \mathbf{X}_0^{[2]} + \mathbb{P}^3_2 \mathbf{X}_0^{[3]} + \mathbb{P}^4_2 \mathbf{X}_0^{[4]}}{1 + \mathbf{Q}^*_1 \mathbf{X}_0}. \qquad (5)$$

where
$$Q_0 = 1, \quad \mathbf{Q}_1 = -(t-t_0)\begin{pmatrix} a \\ 0 \end{pmatrix}, \quad \mathbb{P}^2_2 = a(t-t_0)\begin{pmatrix} 0 & 0 & 0 \\ -b & 3a & 0 \end{pmatrix},$$

$$\mathbb{P}^3_2 = a(t-t_0)^2\begin{pmatrix} 0 & 0 & 0 & 0 \\ -b & 3a & 0 & 0 \end{pmatrix}, \quad \mathbb{P}^4_2 = \frac{a^2(t-t_0)^3}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -b & 3a & 0 & 0 & 0 \end{pmatrix}.$$

We should note that (5) is exact solution of the equation system (4)!

# Energy conservation

It is known that in general cases the symplecticity of the map (exact or approximate map) does not guarantee the energy conservation. That is why we should additionally constrain the used approximated map. In another words on the every step we must guarantee the energy conservation low, which can be written in the following forms

$$E(\mathbf{Q},\mathbf{P},t_k) = E(\mathbf{Q},\mathbf{P},t_{k-1}),\ \forall k \geq 1, \mathbf{X} = \begin{pmatrix}\mathbf{Q}\\ \mathbf{P}\end{pmatrix}, \qquad \mathcal{M}(t_k|t_{k-1}) \circ E(\mathbf{X},t_k) \equiv E(\mathbf{X},t_{k-1}).$$

These conditions can be realized using some correction procedure. We demonstrate this process using the matrix formalism

$$\mathcal{M}(t_k|t_{k-1}) \circ E(\mathbf{X},t_k) = E(\mathcal{M}(t_k|t_{k-1}) \circ \mathbf{X}, t_{k-1})) = E\left(\sum^{\infty} \mathbb{R}^{[1j]}(t_k|t_{k-1})\mathbf{X}^{[j]}\right)$$

For linear case we have $E(\mathbf{Q}_{k-1},\mathbf{P}_{k-1},t_{k-1}) = \frac{1}{2}\left(\mathbf{X}^{\mathrm{T}}(t_{k-1}) \cdot \mathbb{A} \cdot \mathbf{X}_{k-1}(t_{k-1}))\right), \mathbb{A} = \begin{pmatrix}0 & 1\\ -\omega^2 & 0\end{pmatrix}$

$$E\left(\mathbb{R}^{[11]}(t_k|t_{k-1})\mathbf{X}_{k-1}\right) = \left(\mathbf{X}^{\mathrm{T}}_{k-1}\left(\mathbb{R}^{[11]}_k\right)^{\mathrm{T}} \cdot \mathbb{A} \cdot \mathbb{R}^{[11]}_k \mathbf{X}_{k-1}\right) = \left(\mathbf{X}^{\mathrm{T}}_k \cdot \mathbb{A} \cdot \mathbf{X}_k\right) = \left(\mathbf{X}^{\mathrm{T}}_{k-1} \cdot \mathbb{A} \cdot \mathbf{X}_{k-1}\right)$$

where $\mathbb{R}^{[11]}_k = \mathbb{R}^{[11]}(t_k|t_{k-1})$ , and we have $E_k = E_{k-1}$ **!**

# Energy conservation

Similar evaluation for nonlinear Hamiltonian and using full "matrix map" $\sum_{j=1}^{\infty} \mathbb{R}^{[1j]}(t_k|t_{k-1})\mathbf{X}^{[j]}$ leads us to the same result. On the practice we apply some truncated transformation of *N*-th order $\sum_{j=1}^{N} \mathbb{R}^{[1j]}(t_k|t_{k-1})\mathbf{X}^{[j]}$ and similar transformation doesn't conserve nonlinear Hamiltonian!

**There is a problem**: Can we construct an integration scheme that is both symplectic and energy-conserving properties for a broad class of Hamiltonian systems? The well known Zhang and Marsden theorem answer – in general case – **NO**!

If we want to conserve nonlinear Hamiltonian, than we should "correct a little" our truncated matrix map. In another words, some elements of $\mathbb{R}^{[1j]}(t_k|t_{k-1})$ we should be corrected.

For this purpose we can evaluate some equations (see, an example, the correction procedure for symplectification). Here there are some different approaches. The choice of appropriate variant depends on the practical problem: the symplectification condition is **universal property**, while the energy conservation **depends on the energy function** (Hamiltonian)!

# Conclusion (physical problems - 1)

1. The **results are correspond to the existing experimental data.**
2. The matrix formalism can be used for different models of beam dynamics (in the frame of the successive approximations approach), **including space charge forces**.
3. The matrix formalism **can be symplectified** with comparative ease. Also we can **compute approximate invariants** for particle beams.
4. The matrix formalism permits different correction procedures **for energy conservation.**

# Conclusion (mathematical problems - 1)

1. The basic principal difference the matrix formalism for presentation of motion equations in the form of ODE's or Hamiltonian equations: we handle with **two dimensional matrices** instead of multidimensional tensors, similar in MAD, Transport, COSY Infinity and so on.
2. The "improvement" of corresponding models is realized using step-by-step process (using **increasing of approximation order and variation of corresponding matrices**).
3. Linear and "nonlinear" matrices can be evaluated both in **symbolic** (and to keep in special data bases) and in **numerical forms** (using appropriate numerical methods, for example, symplectic Runge-Kutta method or others for corresponding matrix ODE's).

# Conclusion (computational problems - 1)

1. The matrix formalism is **compatible with optimization procedures** of beam dynamics. For this purpose we can use only corresponding matrix elements.
2. The matrix formalism admit **parallelization** and **distribution** procedures (including Grid- and Cloud technologies) naturally.
3. The matrix formalism can be easily embedded into the **Virtual Accelerator** concept.

# Some of corresponding papers (in English)


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