ON A NEW APPROACH FOR DESCRIPTION OF SELF-CONSISTENT DISTRIBUTIONS FOR A CHARGED PARTICLE BEAM

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Abstract

The present report is concerned with the problem of particle phase space distributions for a charged particle beam. A new approach is presented. It provides the possibility to specify various coordinates in the phase space. The main attention has been focused on the case where motion integrals are taken as phase coordinates. Using such coordinates, one can obtain a lot of self-consistent distributions. Some distributions for a breathing beam are considered as examples: generalized Brillouin flow, generalized KV distribution, and others. Besides, this approach allows simple graphical representation of various self-consistent distributions.

PHASE DENSITY

Main feature of the presented approach is covariant description of the particle distribution density in the phase space. The phase space particle distribution is described by a tensor density instead of the scalar distribution function. It allows specifying various coordinates in the phase space. This concept was previously formulated in the works [1, 2].

Let us consider a charged particle beam as a continuous media that occupies an open set in the phase space M. Such distribution are nondegenrate, and this cases can be regarded as most general. According to this model, particle number in an open subregion G, $G \subset M$, is a real number. Call the differential form $n(t,q)$ of degree $m = \dim M$ such that integration of the form over each open set G gives particle number in G the particle distribution density in the phase space, or the phase density:

$$\int\limits_G n = N_G.$$

Here q and t denote position in the phase space and the time correspondingly. The boundaries of G and the form n are assumed sufficiently smooth for integration being possible. Such tensor density has the following physical sense. If we take a cell in the phase space defined by m displacement vector, the density as a polylinear form acting on these displacement vectors gives us a number of particles in this cell.

Consider another case when particle are distributed on an oriented surface S in the phase space that can move, $\dim S = p$, $0 < p < m$. Call the differential form $n(t,q)$ of degree p defined on the surface S such that for any open set G, $G \subset M$,

$$\int\limits_{G \cap S} n = N_G$$

the particle distribution density for this case. This form depends on orientation of the surface. The orientation is defined by an ordered set of $m - p$ vectors. For example, orientation of a two-dimensional surface in the three dimensional space is defined by a vector, and in the four-dimensional and orientation of A change of the orientation can result in change of sign of the form components [3]. Assume that form n and the surface S are also sufficiently smooth for integration being possible.

At last, consider the case of a collection of dicrete particles. Define the scalar function

$$\delta_{q'}(q) = \begin{cases} 1, & q = q', \\ 0, & q \neq q'. \end{cases} \tag{1}$$

If q' depends on t, then this function is also function of t. All functions which values are nonzero only in finite set of points can be represented as linear combination of the functions of form (1). Restrict ourselves only to combinations with all coefficients equal to 1:

$$n(t,q) = \sum_{i=1}^{N} \delta_{q_{(i)}}(q), \qquad q_{(i)} \neq q_{(j)}, \quad \text{if} \quad i \neq j. \tag{2}$$

In this class of functions, define an operation of taking sum of function values in all points $q_{(i)}$, where the function value is nonzero:

$$\sum_{q \in G} n(t,q) \equiv \sum_{i:\, q_{(i)} \in G} n(t, q_{(i)}). \tag{3}$$

Operation defined by equation (3) is analogous to integration of the form of higher degree over G. A scalar function can be regarded as the differential form of degree 0. Therefore, equation (3) set a rule of integration of a form of degree 0 over open set G. As previously, call function of form (2) the phase density for system of pointlike particles if

$$\sum_{q \in G} n(t,q) = N_G.$$

It is easy to understand that the phase density is given by equality (2), where $q_{(i)}$ are positions of the particles in the phase space, $i = \overline{1,N}$, N is the total number of particles in the ensemble.

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At each instant of time t, particle dynamics equations define a vector field $f(t,q)$, which depends on the force acting on a particle.

Assume that for each t $q \in M$ there exists a unique integral line passing through point q. For example, if components of $f(t,q)$ in Cartesian coordinates are continuously differentiable with respect to coordinates and the time, it will be so. The time can be taken as a parameter for the integral lines.

In all three cases, the Vlasov equation can be written in the form

$$n(t+\delta t, F_{f\,\delta t} q) = F_{f\,\delta t} n(t,q). \tag{4}$$

Here $F_{f\,\delta t}$ denotes the operation of Lie dragging of a point or a tensor along the vector field f by the parameter increment δt.

When moving particles always lie on the same surface, that is each points of this surface belongs to the support of this distribution, equation (4) can be rewritten with use of the Lie derivative of the phase density along the vector field f, which is denoted by $\mathfrak{L}_f n(t,q)$, as follows

$$\partial n/\partial t = -\mathfrak{L}_f n(t,q). \tag{5}$$

SPACE OF INTEGRALS OF MOTION

Consider stationary azimuthally symmetric beam in longitudinal magnetic field in which all particles have the same longitudinal velocity v^z [4-16]. Let radius of the beam cross-section R and longituninal component of the magnetic field $B_z(z)$ slow change along beam axis: $\partial B_z/dz \ll B_z/R$. Assume also that the spatial density is uniform within each cross-section: $\varrho = \varrho_0(z)$, $r < R$.

Under the assumptions the vector potential corresponding to the external magnetic field can be taken as

$$A_0 = A_r = A_z = 0, \quad A_\varphi = -cB_z(z)r^2/2,$$

and the vector potential of the self field of the beam as

$$A = (-U(r,z)/c, 0, 0, \beta U(r,z)/c).$$

Here r, φ, z denotes the cylindrical spatial coordinates, $\beta = v_z/c$ is reduced longitudinal velocity, $U(r,z)$ – potential of the self electric field.

Then equation of the azimuthal motion can be written in the form [3]

$$\frac{dp_\varphi}{ds} = \frac{e}{c}\frac{\partial A_j}{\partial \varphi} u^j, \tag{6}$$

and its integral in the form (6)

$$M = r^2(\varphi' + \omega_0). \tag{7}$$

Here p, u denote four-dimensional momentum and velocity of the particle, s -relativistic interval, $\omega_0 = eB_z(z)/2mc$, $M = p_\varphi/mc$, e and m charge and mass of the particle, stroke denotes differentiation with respect to s, summation over j is meant according the Einstein rule.

Equation of the radial motion has the form

$$\frac{dp_r}{ds} = -mcr\varphi'^2 + \frac{e}{\gamma c}\frac{\partial U}{\partial r} - eB_z r\varphi'. \tag{8}$$

Substituting $\varphi' = -\omega_0 + M/r^2$, $p_r = mc\,dr/ds$, and taking into account uniformity of the spatial distribution inside beam cross-section, rewrite the equation (8) in the form

$$\frac{d^2r}{ds^2} = -\omega_0^2 r + \frac{\lambda}{R(z)^2} r + \frac{M^2}{r^3} = -\omega^2 r + \frac{M^2}{r^3}. \tag{9}$$

Here

$$\omega^2 = \omega_0^2 - \frac{\lambda}{R(z)^2}, \quad \lambda = \frac{J}{J_0}\cdot\frac{1}{\beta\gamma}, \quad J_0 = \frac{2\pi\varepsilon_0 mc^3}{e},$$

J is the beam current, $\gamma = (1-\beta^2)^{-1/2}$ is reduced energy, ε_0 is electric constant.

Assume that the beam envelope $R(z)$ is determined only by particles with $M = 0$. Then the equation (9) of the radial motion can be represented in the form:

$$X' = AX, \tag{10}$$

where

$$X = \begin{pmatrix} r \\ r' \end{pmatrix}, \; A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}.$$

Assume that in initial cross-section $z = z_0$ particle fill the ellipse

$$X_0^* B_0 X_0 \le 1, \; B_0 = \begin{pmatrix} a_0^{-2} & 0 \\ 0 & c_0^{-2} \end{pmatrix}.$$

Then at $z \ge z_0$, they will fill ellipses $X^*BX \le 1$, where $B = F^{*-1} B_0 F^{-1}$, and F — matrizant of the system (10).

It is easy to see that

$$R^2 = (B^{-1})_{11} = a_0^{-2} F_{11}^{-2} + c_0^{-2} F_{12}^2.$$

Matrix elements F_{11} and F_{12} satisfy to the equations

$$F_{11}'' = -\omega^2 F_{11}, \; F_{12}'' = -\omega^2 F_{12}.$$

Integrating them, we get the envelope equation in the form

$$R'' = -\omega^2 R + \frac{a_0^2 c_0^2}{R^3}. \tag{11}$$

The system of equation (9), (11) can be reduced to a particular case of the generalized Ermakov system considered in the work [17]. It can be shown that

$$I = (Rr' - rR')^2 + \frac{M^2R^2}{r^2} + \frac{a_0^2c_0^2r^2}{R^2} =$$

$$(\frac{dq}{d\tau})^2 + \frac{M^2}{q^2} + a_0^2c_0^2q^2 \tag{12}$$

is integral of the motion. Here $q = r/R$, $d\tau = ds/R^2$. Integral (12) was introduced for the first time in the work [5]. When $M = 0$, $\omega = \omega(s)$ integral (12) coincides with

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well known Courant-Snyder invariant [18], which is integral for the Ermakov system [19] and for its generalization [17]when $M = 0$, $\lambda \neq 0$.

Call the space with coordinates M and I the space of integrals of motion. It is easy to see that the set of admissible values of M and I in this space is determined by inequalities

$$2a_0c_0|M| < I \leq M^2 + a_0^2c_0^2. \quad (13)$$

Denote this set by $\tilde{\Omega}_1$.

Consider a particle distribution of some thin slice moving along beam axis. The phase space is four-dimensional, and integrals M, I, azimuthal angle φ and particle phase on the trajectory θ can be taken as coordinates in it. Assume that particle uniformly distributed on θ and φ. Under this condition any distribution in the phase space is uniquely defined by distribution in the space of integrals of motion.

There exists a condition which should be satisfied for distribution specified in this way. Such distribution should be spatially uniform in the beam cross-section, as the integral I was obtained under this assumption.

Therefore, we should substitute a density specified in the space of integrals of motion in some integral equation.

There is a more simple way to construct new self-consistent distribution. If we know various distributions uniform inside beam cross-section, we can take their linear combination, and it will be also uniform. If a class of such distributions depends on a parameter, integrating over this parameter also gives spatially uniform distribution.

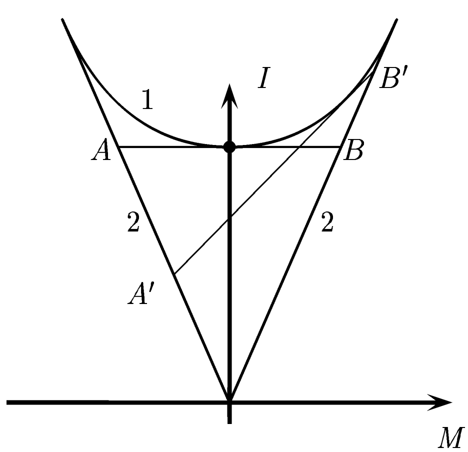

Figure 1: The set of admissible values $\tilde{\Omega}_1$ in the space of integrals of motion for a uniformly charged beam (thick lines).

SELF-CONSISTENT DISTRIBUTIONS

At first, consider a case when particles are distributed on the two-dimensional surface $M = 0$, $I = 0$. In this case the Vlasov equation (5) takes the form

$$\frac{\partial n_{\theta\varphi}}{\partial t} + \dot{\theta}\frac{\partial n_{\theta\varphi}}{\partial \theta} + \dot{\varphi}\frac{\partial n_{\theta\varphi}}{\partial \varphi} = 0.$$

The first term in the left hand side is zero as the distribution is stationary. According to the previous assumption, the second and the third terms are equal to zero. Therefore such distribution is satisfied to the Vlasov equation.

Passing to the Cartesian spatial coordinates x, y, it is easy to find that spatial distribution is uniform inside the beam cross-section. Such distribution is a generalization of the well known Brillouin flow [20].

Consider also a distribution when all particles are uniformly distributed on the segment S_k, which is tangent to upper boundary of the set $\tilde{\Omega}_1$:

$$S_k : I = kM + I_0, \quad I_0(k) = a_0^2c_0^2 - k^2/4,$$

$|k| < 2a_0c_0$, $(M, I) \in \tilde{\Omega}_1$ (segment $A'B'$ on Fig.1). Describe the particle density in the space of the integrals of motion by the differential form of the first degree $f_0 dM$, $f_0 > 0$. In the initial four-dimensional phase space such density is described by the form of degree 3 defined on the segment S_k.

As for each M, I particles are uniformely distributed on θ and φ,

$$n_{\varphi\theta M} = f_0/4\pi P(M, I),$$

where $P(M, I)$ is change of the phase θ along a half of trajectory:

$$P(M, I) = \int\limits_{q_{\min(M,I)}}^{q_{\max(M,I)}} (I - \frac{M^2}{q^2} - a_0^2c_0^2q^2)^{1/2}\, dq = \frac{\pi}{2a_0c_0}.$$

Passing to the Cartesian coordinates $\tilde{x} = x/R$, $\tilde{y} = y/R$, we get

$$n_{\tilde{x}\tilde{y}M} = \frac{n_{\varphi\theta M}}{q|\dot{q}|}.$$

For spatial density we obtain

$$\varrho_{\tilde{x}\tilde{y}M} = \int\limits_{M_1}^{M_2} n_{\tilde{x}\tilde{y}M}\, dM = \frac{a_0c_0f_0}{\pi} = const.$$

Therefore, for such distributions the particles are uniformely distributed inside the beam cross-section.

When $k = 0$ (segment AB on Fig.1), we have analogue of the Kapchinsky-Vladimirsky distribution for nonuniform beam. It is easy to understand that taking a linear combination of such distributions with various k we also get a solution of the Vlasov equation.

As it was mentioned before, various self-consistent distributions can be also found from the integral equation This equation can be written if we express spatial density through the density in the space of integral of motion and equate it to the density for which integrals are gotten.

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