ON THE MINIMAX PROBLEM OF BEAM DYNAMICS OPTIMIZATION

M. Mizintseva*, D. Ovsyannikov
Saint Petersburg State University, Universitetskaya nab. 7/9, St. Petersburg, Russia

Abstract

The problem of simultaneous optimization of the ensemble of trajectories and some selected trajectory arises in the research of the charged particle beam dynamics [1–8].
The present work suggests the use of a smooth functional for the evaluation of the selected trajectories and a minimax functional for the evaluation of the dynamics of the beam of trajectories. A combination of those functionals is considered.

INTRODUCTION

In the present work a new approach to the beam dynamics optimization, based on the use of smooth and non-smooth functionals for the evaluation of the dynamics of the charged particles, is developed. The problem of simultaneous optimization of the program motion and the ensemble of trajectories is formulated. The dynamics of the program motion is evaluated using a smooth integral functional and the dynamics of the ensemble of disturbed motions is evaluated using a non-smooth functional.
In this paper the analytical form of the variation for the combination of a smooth and non-smooth functionals is presented, allowing to develop various methods of optimization. Those methods can be implemented, for instance, to the optimization of particle dynamics in a RFQ structure. It should be noted that the problems of analysis and optimization of the particle dynamics in RFQ accelerators in an equivalent running wave were explored in numerous works [9–14], but those did not utilize non-smooth functionals.

MATHEMATICAL MODEL

Let us consider the following system of differential equations

$$\frac{dx}{dt} = f(t,x,u), \qquad x(0) = x_0. \tag{1}$$

Here $t \in [0,T]$ — independent variable, $T > 0$ is a fixed moment of time; x — n–dimensional phase-vector; $u = u(t)$ — r–dimentional piecewise continuous control vector-function from a class D; $f(t,x,u)$ — n–dimentional reasonably smooth vector-function . Let us call the solution of system (1) a program motion.
At the same time we consider the so-called disturbed motions, which are the solutions of the following system of equations [1]

$$\frac{dy}{dt} = F(t,x,y,u), \qquad y(0) = y_0 \in M_0. \tag{2}$$

Here y — n–dimensional phase-vector; $F(t,x,y,u)$ — n-dimensional reasonably smooth vector-function; M_0 — a compact set.
The trajectories of system (2) are vector-functions $y = y(t,x(t,x_0,u),y_0,u)$, continuously dependent on the program motion $x(t,x_0,u)$ and initial conditions $y_0 \in M_0$.
Let us introduce the set of terminal positions of the system (2)

$$Y = \{y(T,x_0,y_0,u) \mid u \in D, x(0) = x_0, y_0 \in M_0\}.$$

On the solutions of system (1) let us introduce a functional

$$I_1(u) = \int_0^T \varphi_1(x(t,x_0,u))dt + g(x(T))$$

and on the trajectories of system (2) the following functional

$$I_2(u) = \max_{y_T \in Y} \varphi_2(Y).$$

Here φ_1 and φ_2 are non-negative smooth functions.
In the present paper the following functional is studied

$$I(u) = I_1(u) + I_2(u).$$

VARIATION OF THE FUNCTIONAL

Let us consider a variation of the control function $\Delta u(t)$, so that $\tilde{u}(t) = u(t) + \Delta u(t) \in D$.
Let us introduce a set $R_T(u)$, dependent on the control $u = u(t)$ and defined by expression

$$\begin{aligned} R_T(u) = \{\bar{y}_0 : \bar{y}_0 \in M_0, \varphi_2(y(T,x_0,\bar{y}_0,u)) = \\ = \max_{y_0 \in M_0} \varphi_2(y(T,x_0,y_0,u))\}. \end{aligned} \tag{3}$$

Following the logic of [10] lemma can be proved.
Lemma *Let us consider sets $R_T(u)$ and $R_T(\tilde{u})$, defined by the relations (3), corresponding to the allowed controls $u(t)$ and $\tilde{u}(t)$, then*

$$\max_{y_0'' \in R_T(\tilde{u})} \min_{y_0' \in R_T(u)} \|y_0'' - y_0'\| \to 0 \quad \text{when} \quad \|\Delta u\|_L \to 0.$$

The variations equations correspoding to the systems (1 –2) are as follows

$$\begin{aligned} &\frac{d\delta x}{dt} = \frac{\partial f(t,x,u)}{\partial x}\delta x + \Delta_u f(t,x,u), \\ &\delta x(0) = 0; \\ &\frac{d\delta y}{dt} = \frac{\partial F(t,x,y,u)}{\partial x}\delta x + \frac{\partial F(t,x,y,u)}{\partial y}\delta y + \\ &\qquad + \Delta_u F(t,x,y,u), \\ &\delta y(0) = 0. \end{aligned}$$

* m.mizintseva@spbu.ru

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Here

$$\Delta_u f(t,x,u) = f(t,x,u+\Delta u) - f(t,x,u),$$
$$\Delta_u F(t,x,y,u) = F(t,x,y,u+\Delta u) - F(t,x,y,u).$$

Function $\varphi_2(y(T,x_0,y_0,\tilde{u})$ can be represented in the following form

$$\begin{aligned}&\varphi_2(y(T,x_0,y_0,\tilde{u}) + \Delta y(T,x_0,y_0)) =\\ &= \varphi_2(y(T,x_0,y_0,u)) + \frac{\partial \varphi_2(y(T,x_0,y_0,u))}{\partial y}\delta y +\\ &\qquad + o(\|\Delta y(T)\|_C).\end{aligned} \quad (4)$$

Using (4) the variation of the functional $I_2(u)$ can be obtained as follows

$$\delta I_2 = \max_{y_0 \in R_T(u)} \frac{\partial \varphi_2(y(T,x_0,y_0,u))}{\partial y}\delta y(T).$$

The variation of the functional represented by a smooth function is [1, 3]

$$\delta I_1 = \int_0^T \frac{\partial \varphi_1(x(t,x_0,u))}{\partial x}\delta x dt + \frac{\partial g(x(T))}{\partial x}\delta x(T).$$

Then the variation of the functional $I(u)$ is

$$\delta I = \delta I_1 + \delta I_2.$$

Let us introduce functions ψ and λ

$$\begin{aligned}&\psi^{*'} + \psi^* \frac{\partial f}{\partial x} = \frac{\partial \varphi_1}{\partial x} - \lambda^* \frac{\partial F}{\partial x},\\ &\psi^*(T) = -\frac{\partial g(x(T))}{\partial x},\\ &\lambda^{*'} + \lambda^* \frac{\partial F}{\partial y} = 0,\\ &\lambda^*(T) = -\frac{\partial \varphi_2(Y)}{\partial y}.\end{aligned}$$

Then the variation of the functional can be written as follows

$$\delta I(u) = \max_{y_0 \in R_T(u)} \int_0^T (\psi^* \Delta_u f(t,x,u) - \lambda^* \Delta_u F(t,x,y,u))dt.$$

Let us introduce Hamilton's function

$$H(t,x,u,\psi,\lambda,u) = \psi^* f(t,x,u) + \lambda^* F(t,x,y,u),$$

then variation will be

$$\begin{aligned}\delta I(u) = \max_{y_0 \in R_T(u)} \int_0^T \big(&H(t,x,y,\psi,\lambda,u) -\\ &- H(t,x,y,\psi,\lambda,\tilde{u})\big)dt.\end{aligned}$$

The obtained expression for the variation of the functional can be applied to various problems of optimization in electro-physical devices.

BEAM DYNAMICS IN A RFQ STRUCTURE

Approach using systems of differential equations (1)–(2) can be applied to the modeling of beam dynamics in a RFQ accelerator in an equivalent running wave. The dynamics of a synchronous particle is described by the following equations [13, 14]

$$\begin{aligned}&\frac{d\gamma_s}{dz} = u_1 \frac{qU}{2m_0c^2} cosu_2,\\ &\gamma_s(0) = \gamma_{s0}.\end{aligned}$$

Disturbed motions are presented by the deviations in phase $\psi = \varphi - \varphi_s$ and reduced energy $p_\psi = \gamma - \gamma_s$ from the synchronous particle

$$\begin{aligned}&\frac{dp_\psi}{dz} = u_1 \frac{qU}{2m_0c^2}(cosu_2 - cos(\psi + u_2)),\\ &p_\psi(0) = p_{\psi 0} \in M_0,\\ &\frac{d\psi}{dz} = \frac{2\pi}{(\gamma_s^2 - 1)^{3/2}} p_\psi,\\ &\psi(0) = \psi_0 \in M_0.\end{aligned}$$

This model quite accurately describes the dynamics of the particle beam.
Figure 1 below shows the acceleration intensity. Figure 2 shows the phase of the synchronous particle.

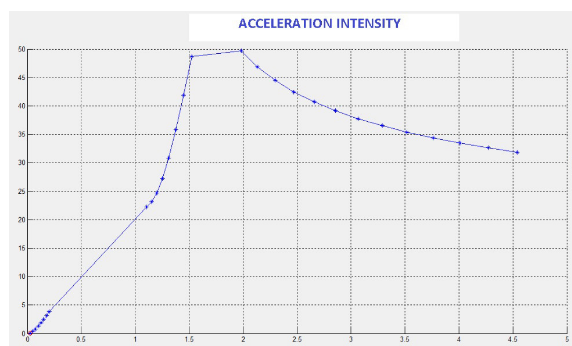

Figure 1: Acceleration intensity.

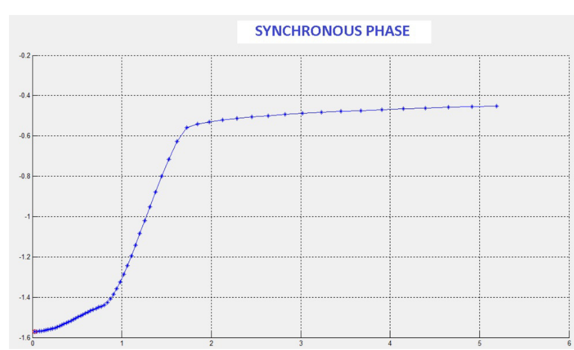

Figure 2: Phase of the synchronous particle.

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Figure 3 shows reduced energy deviations from the synchronous particle. Figure 4 shows phase deviations from the synchromous particle.

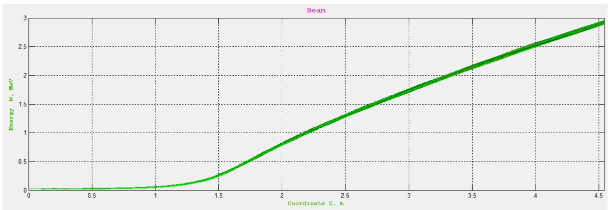

Figure 3: Beam reduced energy.

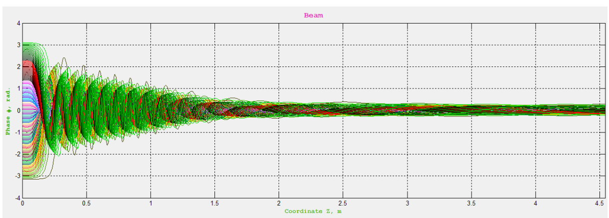

Figure 4: Deviations from the synchronous phase.

The modeling data is pretty good, but can be futher improved by implementing the proposed appoach to optimization.
The following functionals can be introduced.
Let us introduce a smooth functional $I_1(u)$

$$I_1(u) = \int_0^L \varphi_1(A_{def})dz + g(x(L)).$$

Here A_{def} — is the defocusing factor, $g(x(L)) = (\gamma_s(L) - \tilde{\gamma}(L))^2$ — in this case the aim of the optimization is the minimization of the defocusing factor and evaluation of the deviation of the reduced energy of the synchronous particle from a fixed value $\tilde{\gamma}$ at the end of the accelerating structure.
The non-smooth functional $I_2(u)$ can evaluate the maximum deviation of particles in phase and reduced energy from the synchronous particle. In paticular, the following functional can be introduced

$$I_2(u) = \max_{y_T \in Y} \psi^2.$$

Resulting functional $I(u) = I_1(u) + I_2(u)$ allows to consider simultaneously the program motion and the disturbed motions in the problem of optimal control.

CONCLUSION

The proposed approach to simultaneous optimization of program and disturbed motions looks very promising in problems where it is important not just to evaluate the process in general, but also to take into account the worst, the most deviating particles.
The obtained variation of the functional can be used for construction of directed methods of minimization.

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