ON STABILIZATION OF SYSTEMS OF LINEAR EQUATIONS WITH LINEAR INCREASING TIME DELAY BY OBSERVATION

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Abstract

In this paper we investigate a possibility of the linear differential system stabilization with time proportional delay by the linear observation. Using the sufficient conditions of asymptotic stability for the linear systems with linearly increasing delay we obtain some conditions of the asymptotic evaluation system existence for the original system. Then we use the asymptotic evaluation system for the construction of the stabilizing control and derive the sufficient conditions for the existence of such control.

INTRODUCTION

Differential-difference equations with time delay are often used in mathematical models describing the dynamics of beams of the charged particles. For example linear equation of the second order with a constant time delay describes in the smoothed approach dynamics of a beam of the charged particles in synchrotrons with a feedback system [1]. However the time delay cannot always be considered constant. The time proportional delay can occur at acceleration of beams of the charged particles in the cyclotron. It should be noted that linear increasing time delay is unbounded and well known approaches are not applicable for stability analysis such systems.

The stabilizing control for the system of linear equations could be constructed by the information on a state vector of the system. Sometimes the state vector is unknown but we know some linear combinations of its components. Then there is a problem on construction of the stabilizing control with incomplete information.

Let us consider the following linear system

$$\dot{x} = A_0 x(t) + A_1 x(\alpha t) + Bu \tag{1}$$

$$y = Kx(t)\,. \tag{2}$$

Here x is n-dimensional state vector; u is r-dimensional control vector; y is scalar output; A_0, A_1, B, K are given real matrices $n\times n$; $n\times r$ and $1\times n$; $0<\alpha<1$. We must construct the control $u = u\big(x(t); x(\alpha t)\big)$ using output (2).

One of the basic methods of solving such problem is the construction of the asymptotic evaluation system [2]. Some sufficient conditions of existence of this system for n scalar outputs $y_i = Rx(t-ih); i = 1;\ldots;n$ are presented in [3].

The aim of this paper is to obtain the conditions for the matrices A_0, A_1, B, K under which the stabilizing control may be constructed by output (2). The main results of the paper are construction of the asymptotic evaluation system using the output (2) and construction of the stabilizing control for the system (1).

The paper is organized as follows. The next section contains the investigation of an auxiliary system of linear differential equations. The structure of the asymptotic evaluation system for system (1) is deduced in this section. The main result is presented in section 3. A theorem about the sufficient conditions for existence of the stabilizing control is proved here. Section 4 contains a numerical example on construction of the stabilizing control.

AUXILIARY RESULT

Definition [2]. A system

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - K\hat{x})$$

is known as asymptotic evaluation system of a system without time delay

$$\dot{x} = Ax + Bu$$
$$y = Kx$$

if matrix L can be chosen such that for any $x(0); \hat{x}(0)$ the following condition is satisfied:

$$\hat{x}(t) - x(t) \to 0 \quad \text{as} \quad t\to+\infty\,. \tag{3}$$

We consider a time delay system of the form

$$\begin{aligned}\dot{\hat{x}} &= A_0\hat{x}(t) + A_1\hat{x}(\alpha t) + Bu + L_0\big(y - K\hat{x}(t)\big)\\ &+ L_1\big(y(\alpha t) - K\hat{x}(\alpha t)\big)\end{aligned} \tag{4}$$

Here $L_0; L_1$ are unknown constant real vectors. Now we introduce two matrices:

$S_0 = \left(K^T; A_0^T K^T;\ldots;\left(A_0^T\right)^{n-1} K^T\right)$ and

$S_1 = \left(K^T; A_1^T K^T;\ldots;\left(A_1^T\right)^{n-1} K^T\right)$.

Theorem 1. If $rang S_0 = rang S_1 = n$ then vectors L_0 and L_1 of system (4) can be chosen so that condition (3) satisfied for any $x(0), \hat{x}(0)$.

Proof. Let $z(t) = \hat{x}(t) - x(t)$ where $x(t)$ is a solution of the system (1) and $\hat{x}(t)$ is a solution of the system (4). Then

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$$\begin{aligned}
\dot{z}(t) &= \dot{\hat{x}}(t) - \dot{x}(t) = A_0\hat{x}(t) + A_1\hat{x}(\alpha t) + Bu + \\
& L_0\left(y(t) - K\hat{x}(t)\right) + L_1\left(y(\alpha t) - K\hat{x}(\alpha t)\right) - \\
& - A_0 x(t) - A_1 x(\alpha t) - Bu = \\
& = A_0\left(\hat{x}(t) - x(t)\right) + A_1\left(\hat{x}(\alpha t) - x(\alpha t)\right) + \\
& L_0 K\left(x(t) - \hat{x}(t)\right) + L_1 K\left(x(\alpha t) - \hat{x}(\alpha t)\right) = \\
& = \left(A_0 - L_0 K\right)z(t) + \left(A_1 - L_1 K\right)z(\alpha t).
\end{aligned} \tag{5}$$

So $z(t)$ is a solution of the system

$$\dot{z}(t) = Pz(t) + Qz(\alpha t), \tag{6}$$

where $P = A_0 - L_0 K$ and $Q = A_1 - L_1 K$. We now show that the vectors L_0 and L_1 can be chosen such that $z(t) \to 0$ as $t \to +\infty$.

Let us consider two functions [4]

$$f(\lambda) = \det(\lambda I - P); \quad g(\mu) = \det\left(P + Qe^{\mu \ln\alpha}\right). \tag{7}$$

Here I denotes the identity matrix. Now we introduce two sets: $\Lambda = \{\lambda | f(\lambda) = 0\}$ and $\mathrm{M} = \{\mu | g(\mu) = 0\}$. Then we define the following characteristics: $\overline{\lambda} = \max_{\lambda \in \Lambda} \operatorname{Re}\lambda$ and $\overline{\mu} = \max_{\mu \in \mathrm{M}} \operatorname{Re}\mu$ ($\overline{\mu} = -\infty$ if M is empty). Well known [4] that the system (6) is asymptotically stable if $\overline{\lambda} < 0$ and $\overline{\mu} < 0$.

Since $rangS_0 = n$ then the system

$$\dot{x} = A_0 x(t), \quad y = Kx(t)$$

be completely observable. Then the vector L_0 can be chosen such that the condition $\overline{\lambda} < 0$ is satisfied. Moreover the eigenvalues of matrix P can be any defined in advance. Let the matrix P be constructed. Let us consider the equation

$$\det\left(P + Qe^{\mu \ln\alpha}\right) = 0. \tag{8}$$

Since $0 < \alpha < 1$ then the condition $\overline{\mu} < 0$ be satisfied for $\left|\alpha^{\mu}\right| > 1$. Now we rewrite the equation (8) in the form

$$\det\left(\left(A_0 - L_0 K\right) + \left(A_1 - L_1 K\right)\beta\right) = 0, \tag{9}$$

where $\beta = \alpha^{\mu}$.

Since $\det\left(A_0 - L_0 K\right) \neq 0$ then the equation (9) can be rewriting as

$$\begin{aligned}
&\det\left(I + \left(A_0 - L_0 K\right)^{-1}\left(A_1 - L_1 K\right)\beta\right) = \\
&\det\left(I - \left(A_0 - L_0 K\right)^{-1}\left(L_1 K - A_1\right)\beta\right) = \\
&= \beta^n \det\left(\gamma I - \left(A_0 - L_0 K\right)^{-1}\left(L_1 K - A_1\right)\right) = 0,
\end{aligned} \tag{10}$$

where $\gamma = 1/\beta$.

Let $D = \left(A_0 - L_0 K\right)^{-1}\left(L_1 K - A_1\right)$. Now we observe that the condition $\left|\alpha^{\mu}\right| > 1$ is equivalent to condition $|\gamma| < 1$. This means that the matrix D has to satisfy Schur-Kohn criterion.

On the other hand $rangS_1 = n$ therefore the system

$$\dot{x} = A_1 x(t), \quad y = Kx(t)$$

be completely observable too. Then the vector L_1 can be chosen such that eigenvalues of matrix $L_1 K - A_1$ can be any defined in advance.

Let us consider the characteristic equation of the matrix D. It has a form

$$\det(\lambda I - D) = \lambda^n + d_1\lambda^{n-1} + \cdots + d_n = 0, \tag{11}$$

where d_j $(j = 1, \ldots, n)$ are linear function of the components of the vector L_1. Now we introduce the equation

$$\lambda^n + \nu_1\lambda^{n-1} + \cdots + \nu_n = \prod_{s=1}^{n}\left(\lambda - \lambda_s\right) = 0, \tag{12}$$

where $\left|\lambda_s\right| < 1$ $(s = 1, \ldots, n)$ and the numbers λ_s are chosen in advance. After making the coefficients of the equations (11) and (12) equal we obtain a linear algebraic system of equations $d_j = \nu_j$; $(j = 1, \ldots, n)$. The components of the vector L_1 will solutions of this system.

CONSTRUCTION OF THE STABILIZING CONTROL

Now we introduce two matrices:

$$S_2 = \left(B; A_0 B; \ldots; A_0^{n-1}B\right); \; S_3 = \left(B; A_1 B; \ldots; A_1^{n-1}B\right).$$

Theorem 2. If $rangS_2 = rangS_3 = n$ then the stabilizing control for the system (1) has the form

$$u = C_0\hat{x}(t) + C_1\hat{x}(\alpha t), \tag{13}$$

where $\hat{x}(t)$ is a solution of the system (4).

Proof. Let $rangS_2 = rangS_3 = n$. Then we can construct the matrices C_0 and C_1 such that the system

$$\dot{x}(t) = \left(A_0 + BC_0\right)x(t) + \left(A_1 + BC_1\right)x(\alpha t) \tag{14}$$

will asymptotically stable. Then we prove that the control (13) is the stabilizing control for the system (1). Let us consider the following system

$$\begin{cases}
\dot{x} = A_0 x(t) + A_1 x(\alpha t) + Bu \\
\dot{\hat{x}} = A_0\hat{x}(t) + A_1\hat{x}(\alpha t) + Bu + L_0\left(y(t) - K\hat{x}(t)\right) \\
+ L_1\left(y(\alpha t) - K\hat{x}(\alpha t)\right)
\end{cases} \tag{15}$$

where the control has the form (13). Let the vectors L_0 and L_1 be chosen as in Theorem 1. The closed loop system has the form

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$$\begin{cases}\dot{x} = A_0 x(t) + A_1 x(\alpha t) + BC_0\hat{x}(t) + BC_1\hat{x}(\alpha t)\\ \dot{\hat{x}} = (A_0 + BC_0)\hat{x}(t) + (A_1 + BC_1)\hat{x}(\alpha t) + \\ + L_0(y(t) - K\hat{x}(t)) + L_1(y(\alpha t) - K\hat{x}(\alpha t))\end{cases}. \quad (16)$$

Let $z(t) = \hat{x}(t) - x(t)$ again. Here $x(t)$ is the solution of system (1) and $\hat{x}(t)$ is the solution of system (4). Then system (16) transforms to the form

$$\begin{cases}\dot{x} = (A_0 + BC_0)x(t) + (A_1 + BC_1)x(\alpha t) + \\ + BC_0 z(t) + BC_1 z(\alpha t)\\ \dot{z} = (A_0 - L_0 K)z(t) + (A_1 - L_1 K)z(\alpha t)\end{cases}. \quad (17)$$

Now note that the condition $z(t) \to 0$ as $t \to +\infty$ is satisfies in accordance with Theorem 1. The matrices C_0 and C_1 are chosen such that the system (14) is asymptotically stable. So the system (17) is asymptotically stable too. Therefore, control (13) is the stabilizing control for system (15). But system (1) is a part of system (15) therefore the control (13) is the stabilizing control for system (1).

EXAMPLE

Let the system (1) has a form

$\begin{cases}\dot{x}_1 = 2x_1(t) + x_2(t) + x_1(\alpha t) + x_2(\alpha t)\\ \dot{x}_2 = 4x_2(t) + x_2(\alpha t) + u_2\end{cases}$, and the system (2) has a form $y = x_1(t)$.

Let $L_0 = \begin{pmatrix} l_1^{(0)} \\ l_2^{(0)} \end{pmatrix}$, then $P = \begin{pmatrix} 2 - l_1^{(0)} & 1 \\ -l_2^{(0)} & 4 \end{pmatrix}$.

Therefore

$\det(\lambda I - P) = \lambda^2 + (l_1^{(0)} - 6)\lambda + (8 - 4l_1^{(0)} + l_2^{(0)}) = 0$. Let

$L_0 = \begin{pmatrix} 10 \\ 36 \end{pmatrix}$ then P is Hurwitz matrix and the condition

$\overline{\lambda} < 0$ is satisfied.

Let $L_1 = \begin{pmatrix} l_1^{(1)} \\ l_2^{(1)} \end{pmatrix}$ then $L_1 K - A_1 = \begin{pmatrix} l_1^{(1)} - 1 & -1 \\ l_2^{(1)} & -1 \end{pmatrix}$

and $$D = P^{-1}(L_1 K - A_1) = \begin{pmatrix} l_1^{(1)} - 1 - \dfrac{l_2^{(1)}}{4} & \dfrac{-3}{4} \\ 9(l_1^{(1)} - 1) - 2l_2^{(1)} & -7 \end{pmatrix}.$$

Therefore

$$\det(\lambda I - D) = \lambda^2 + \left(8 - l_1^{(1)} - \frac{l_2^{(1)}}{4}\right)\lambda + \frac{1}{4}\left(1 - l_1^{(1)} + l_2^{(1)}\right) = 0.$$

Now let us consider the equation (12) in a form

$$\left(\lambda - \frac{1}{2}\right)^2 = \lambda^2 - \lambda + \frac{1}{4} = 0.$$

The condition $|\lambda_s| < 1 \quad (s = 1,2)$ is satisfied. The components of the vector L_1 are solutions of this system

$$\begin{cases} 8 - l_1^{(1)} - \dfrac{l_2^{(1)}}{4} = -1 \\ 1 - l_1^{(1)} + l_2^{(1)} = 1 \end{cases}.$$

Therefore $l_1^{(1)} = l_2^{(1)} = 7{,}2$ and $L_1 = \begin{pmatrix} 7{,}2 \\ 7{,}2 \end{pmatrix}$. The vectors C_0 and C_1 may be similarly constructed.

CONCLUSION

The system of differential equations with linear increasing time-delay is investigated in this paper. The sufficient conditions of existence for the corresponding asymptotic evaluation system are obtained. The stabilizing control is constructed.

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