MATHEMATICAL OPTIMIZATION MODEL OF LONGITUDINAL BEAM DYNAMICS IN KLYSTRON-TYPE BUNCHER*

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Abstract

The paper presents recurrent integral-differential beam evolution model. This model is convenient for mathematical description of specific dynamic processes with due account of particle interaction and electric fields excitation by moving beam. On the basis of this model the problem of beam dynamics optimization is formalized as trajectory ensemble control problem. Analytical expression for quality functional gradient is obtained. Theoretical results are applied for solving problem of beam dynamics optimization in klystron-type buncher.

RECURRENT INTEGRAL-DIFFERENTIAL BEAM EVOLUTION MODEL

Let us consider beam dynamics description by recurrent system of integral-differential equations. Finite iteration process is introduced. At every iteration beam evolution is described by the equations of the following form:

$$\frac{d\mathbf{x}^{(k)}}{d\tau} = \mathbf{f}^{(k)}\left(\tau,\mathbf{x}^{(k)},\mathbf{u},\mathbf{H}^{(k-1)}(\mathbf{u})\right) = \mathbf{f}_1\left(\tau,\mathbf{x}^{(k)},\mathbf{u},\mathbf{H}^{(k-1)}(\mathbf{u})\right) + \int\limits_{M_{\tau,\mathbf{u}}^{(k)}} \mathbf{f}_2\left(\tau,\mathbf{x}^{(k)},\mathbf{y}_\tau^{(k)}\right)\rho^{(k)}\left(\tau,\mathbf{y}_\tau^{(k)}\right)d\mathbf{y}_\tau^{(k)}, \tag{1}$$

$$\frac{\partial\rho^{(k)}}{\partial\tau} + \frac{\partial\rho^{(k)}}{\partial\mathbf{x}^{(k)}}\mathbf{f}^{(k)}\left(\tau,\mathbf{x}^{(k)},\mathbf{u},\mathbf{H}^{(k-1)}(\mathbf{u})\right) + \rho^{(k)} div_{\mathbf{x}}\mathbf{f}^{(k)}\left(\tau,\mathbf{x}^{(k)},\mathbf{u},\mathbf{H}^{(k-1)}(\mathbf{u})\right) = 0 \tag{2}$$

with initial conditions

$$\mathbf{x}^{(k)}(0) = \mathbf{x}_0 \in M_0,\ \rho^{(k)}(0,\mathbf{x}) = \rho_0(\mathbf{x}). \tag{3}$$

Here $\tau \in [0,T]$ is independent variable; T is fixed; k is iteration number $(1 \le k \le K)$; $\mathbf{x}^{(k)}$ (or $\mathbf{y}^{(k)}$) is n-vector of phase coordinates; $\mathbf{u}(\tau)$ is r-vector of control; $\mathbf{f}_1\left(\tau,\mathbf{x}^{(k)},\mathbf{u},\mathbf{H}^{(k-1)}(\mathbf{u})\right)$ and $\mathbf{f}_2\left(\tau,\mathbf{x}^{(k)},\mathbf{y}^{(k)}\right)$ are n-vector functions; $\mathbf{H}^{(k-1)}(\mathbf{u})$ is the matrix containing the values $H_{sj}^{(k-1)}(\mathbf{u})$, $s=\overline{1,S}$, $j=\overline{J_1,J_2}$ of functionals defined on beam trajectories at previous iteration; $\rho^{(k)}\left(\tau,\mathbf{x}^{(k)}\right)$ is phase density corresponding to dynamic system (1); $M_{\tau,\mathbf{u}}^{(k)} = \left\{\mathbf{x}_\tau^{(k)} = \mathbf{x}^{(k)}\left(\tau,\mathbf{x}_0,\mathbf{u}\right) : \mathbf{x}_0 \in M_0\right\}$, M_0 is open bounded initial phase domain; $\rho_0(\mathbf{x})$ is initial phase density; it is supposed $\int\limits_{M_0}\rho_0(\mathbf{x}_0)d\mathbf{x}_0 = 1$.

The components of every matrix $\mathbf{H}^{(l)}$, $l=\overline{1,K}$ are the values of functionals

$$H_{sj}^{(l)}(\mathbf{u}) = \int\limits_0^T \int\limits_{M_{\tau,\mathbf{u}}^{(l)}} C_{sj}(\tau,\mathbf{x}_\tau^{(l)},\overline{\mathbf{x}}^{(l)}(\tau),\mathbf{u})\rho^{(l)}\left(\tau,\mathbf{x}_\tau^{(l)}\right)d\mathbf{x}_\tau^{(l)}d\tau, \tag{4}$$

$s=\overline{1,S}$, $j=\overline{J_1,J_2}$; here $\overline{\mathbf{x}}^{(l)}(\tau) = \int\limits_{M_{\tau,\mathbf{u}}^{(l)}} \mathbf{x}_\tau^{(l)}\rho\left(\tau,\mathbf{x}_\tau^{(l)}\right)d\mathbf{x}_\tau^{(l)}$ is average phase vector at l-th iteration. Note that $\mathbf{H}^{(0)} = \mathbf{0}$.

The resulting beam evolution is to be achieved at the last iteration number K.

Beam evolution model suggested is based on formalization and generalization of iterative method of beam dynamics simulation in floating-drift klystron with due account of Coulomb repulsion and RF fields excitation in resonators. According to this method excited fields are represented via induced current Fourier decomposition [1]. Vector-function $\mathbf{f}_1\left(\tau,\mathbf{x}^{(k)},\mathbf{u},\mathbf{H}^{(k-1)}(\mathbf{u})\right)$ is determined by the method of RF fields description. Functional values (4) represent induced current Fourier harmonics in resonators (excluding modulator) and are used to express excited fields at next $(l+1)$-th iteration and thus to determine vector-function $\mathbf{f}_1\left(\tau,\mathbf{x}^{(l+1)},\mathbf{u},\mathbf{H}^{(l)}(\mathbf{u})\right)$ in dynamics equation (1). Vector function $\mathbf{f}_2(\tau,\mathbf{x},\mathbf{y})$ is defined by particle interaction accounting (in view of space-charge forces representation given in [2]).

It should be noted that model (1)-(4) may be convenient for wide class of beam evolution iterative descriptions with due account of beam dynamics dependence on the functionals defined on beam trajectories at previous iteration. These functionals approximate the fields generated by moving beam itself.

Beam evolution modeling with due account of the fields mentioned (in particular, Coulomb forces) may be performed in different ways [2-14]. Analytical representation of these fields allows to formalize beam dynamics optimization problem and to obtain quality criterion gradient analytical expression [2,11,12].

OPTIMIZATION PROBLEM

Let us introduce quality criterion of the dynamic process (1)-(4) as a function of functionals defined on beam trajectories at finite iteration:

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$$\Gamma(\mathbf{u}) = F(\Gamma_1(\mathbf{u}), \Gamma_2(\mathbf{u}), \ldots, \Gamma_N(\mathbf{u})), \quad (5)$$

$$\Gamma_i(\mathbf{u}) = \int_0^T \int_{M_{\tau,\mathbf{u}}^{(K)}} \Phi_i\left(\tau, \mathbf{x}^{(K)}, \mathbf{u}, \mathbf{A}_i(\mathbf{u})\right) \rho^{(K)}\left(\tau, \mathbf{x}_\tau^{(K)}\right) d\mathbf{x}_\tau^{(K)} d\tau, \quad (6)$$

$i = \overline{1, N}$. Here $F(\Gamma_1, \ldots \Gamma_N)$, $\Phi_i(\tau, \mathbf{x}, \mathbf{u}, \mathbf{A}_i)$, $i = \overline{1, N}$ are smooth functions; $\mathbf{A}_i(\mathbf{u})$, $i = \overline{1, N}$ are M-vectors of values of integral functionals

$$A_{im}(\mathbf{u}) = \int_0^T \int_{M_{\tau,\mathbf{u}}^{(K)}} V_{im}\left(\tau, \mathbf{x}^{(K)}, \mathbf{u}\right) \rho^{(K)}\left(\tau, \mathbf{x}_\tau^{(K)}\right) d\mathbf{x}_\tau^{(K)} d\tau, \quad (7)$$

$i = \overline{1, N}$, $m = \overline{1, M}$, with smooth integrands. The expressions (7) may be interpreted as particle characteristics averaged over the device spatial cross-sections (when τ is the time or its analogue).

So beam dynamics optimization problem is reduced to hill-climbing problem with objective functional (5)-(6). This problem is investigated on the basis of mathematical methods suggested by D.A. Ovsyannikov and widely used for treatment different beam dynamics optimization problems [2,3,7,11,12]. Analytical expression of criterion functional (5)-(6) variation is derived in terms of auxiliary functions satisfying the special equations on dynamic system trajectories. This expression allows to obtain objective functional gradient and to apply directed optimization methods. The results obtained may be used for beam dynamics optimization in different structures.

BEAM DYNAMICS IN BUNCHER: MODELING AND OPTIMIZATION

Model (1)-(4) is used for beam evolution description in klystron buncher. In this case independent variable is $\tau = ct$, where c is the velocity of light, t is the time; $[0, T]$ is independent variable segment sufficiently large for any particle to pass the structure; $n = 2$; $\mathbf{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}) = (z^{(k)}, p^{(k)})$, where z is longitudinal coordinate, p is reduced impulse of electron. RF fields in resonators (except modulator) are simulated on the basis of induced current first harmonics [1, 11];

$$f_{1,1}(\mathbf{x}) = x_2 / \sqrt{1 + x_2^2}\ ;\ S = 2,\ J_1 = 2,\ J_2 = J\ ;$$

$$f_{1,2}(\tau, \mathbf{x}, \mathbf{u}, \mathbf{H}) = -\frac{e}{m_0 c^2}\left[\frac{U_1}{d_1}\widetilde{E}_1(x_1 - \xi_1)\sin(2\pi\tau/\lambda + \Phi_1) + \right.$$

$$\left. + \sum_{s=1}^{2}\sum_{j=2}^{J} B_{sj}(\tau, \mathbf{x}, \mathbf{u}) H_{sj}\right];$$

$$B_{sj}(\tau, \mathbf{x}, \mathbf{u}) = \rho_j Q_j / d_j\, \widetilde{E}_j(x_1 - \xi_j)\cos\theta_j \eta_s(2\pi\tau/\lambda + \theta_j),$$

$$C_{sj}(\tau, \mathbf{x}, \overline{\mathbf{x}}, \mathbf{u}) = \frac{2I}{d_j}\frac{x_2}{\sqrt{1 + x_2^2}}\widetilde{E}_j(x_1 - \xi_j) \times$$

$$\times \Pi_\varepsilon\left(\overline{x}_1 - \xi_j, 0.5\lambda\overline{x}_2 / \sqrt{1 + \overline{x}_2^2}\right)\eta_s(2\pi\tau/\lambda),$$

$$s = \overline{1, 2},\ j = \overline{2, J};$$

$$\eta_1(\xi) = \cos(\xi)\ ;\ \eta_2(\xi) = \sin(\xi)\ .$$

Here J is total number of resonators; e is absolute value of electron charge; m_0 is electron rest-mass; λ is RF field wavelength; U_1 and Φ_1 are prescribed amplitude and initial phase of RF field voltage in first resonator (modulator); j is resonator number; ρ_j and Q_j are wave resistance and Q-factor; d_j is electrical gap length; ξ_j is resonator center coordinate; smooth bell-shaped function $\widetilde{E}_j(\eta)$ presents electric field intensity distribution along the gap axis; θ_j is the mismatch angle; $\theta_j = arctg(2Q_j\, \Delta f_j / f_0)$; $\Delta f_j / f_0$ is resonator mismatch with respect to basic frequency $f_0 = c/\lambda$; $\Pi_\varepsilon(\xi, h)$ is smooth approximation of the step function $\Pi(\xi, h) = U(\xi + h)U(h - \xi)$, where $U(\eta)$ is Heaviside function; $\mathbf{u}$ is the vector of control parameters: $\mathbf{u} = (\Delta f_2 / f_0, \ldots, \Delta f_J / f_0, \xi_2, \ldots, \xi_J, \xi_{ex})$, where ξ_{ex} is device exit coordinate; I is average beam current.

Electron beam is considered to have constant radius R and to move inside the conducting channel of radius a. Model particles are supposed to be disks-clouds (cylinders) with radius R and thickness 2Δ. To describe Coulomb forces we use the analytical expression for electric field intensity [6]. Thus,

$$f_{2,1} = 0\ ;\ f_{2,2}(\tau, \mathbf{x}, \mathbf{y}) = \frac{e}{m_0 c^2}\frac{I\lambda a^2}{2\pi\varepsilon_0 c\Delta^2 R^2} \times$$

$$\times \sum_{m=1}^{\infty}\frac{J_1^2(\mu_m R/a)}{\mu_m^4 J_1^2(\mu_m)} G_m\left(\sqrt{1 + y_2^2}(x_1 - y_1)\right);$$

$$G_m(\eta) = [2g_m(\eta) - g_m(\eta + 2\Delta) - g_m(\eta - 2\Delta)];$$

$$g_m(\eta) = sign(\eta)\left(1 - e^{-\mu_m|\eta|/a}\right),$$

where ε_0 is electric constant; $J_l(\eta)$ is Bessel function of the l-th order; μ_m, $m = 1, 2, \ldots$ are the roots of Bessel function $J_0(\eta)$.

It should be mentioned, that numerical experiments confirm the iteration process convergence. The criterion of iteration process completion is the coincidence (with necessary accuracy) of the corresponding values of functionals (4) on two successive iterations.

Mathematical optimization method presented above is applied for solving problem of bunching efficiency maximization. Quality criterion K_B is defined as the proportion of particles getting required phase domain at buncher exit: $|\varphi - \overline{\varphi}| \le \Delta\varphi$, $|W - \overline{W}| \le bW_0$. Here φ and

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W are correspondingly the phase and kinetic energy of particle at device exit, $\overline{\varphi}$ and $\overline{W}$ are mean values of the quantities mentioned; $\Delta\varphi$ and b are given constants; W_0 is initial energy of particles. This criterion is approximated by the functional of the form (6), where

$$\Phi(\tau, \mathbf{x}, \mathbf{u}, \mathbf{A}) =$$

$$= \Pi_\varepsilon\left(\frac{2\pi\tau}{\lambda} - A_1, \Delta\varphi\right)\Pi_\varepsilon\left(\sqrt{1+x_2^2} - A_2, b\gamma_0\right)S_\varepsilon(x_1 - \xi_{ex}),$$

$$S_\varepsilon(\xi) = \frac{1}{2\varepsilon}\begin{cases}\cos(\pi\xi/\varepsilon)+1, & |\xi| < \varepsilon \\ 0, & |\xi| \ge \varepsilon\end{cases},$$

$A_i(\mathbf{u})$ are the functionals of the form (7) introduced to approximate average values of phase and energy at buncher exit; ε is sufficiently small constant to provide required approximation accuracy. Quality functional gradient with respect to parameters (mismatches and positions of resonators) is obtained.

Gradient optimization of device parameters is performed for klystron buncher with following main characteristics: $W_0 = 500\,\text{keV}$; $I = 10\,\text{A}$; $a = 0.006\,\text{m}$; $R = 0.003\,\text{m}$; $\lambda = 0.125\,\text{m}$; input power $P_{ent} = 1\,\text{kVt}$. The device contains four resonators with electric gap length 0.027 m. The required phase and energy intervals at buncher exit are as follows: $|\varphi - \overline{\varphi}| \le 2\pi/9$, $|W - \overline{W}| \le 0.1W_0$. After the optimization bunching efficiency K_B increased from 0.34 until 0.67.

The numerical experiments performed confirm the efficiency of developed mathematical methods of beam dynamics modeling and optimization. It should be noted, that these methods may be successfully applied for investigation of longitudinal and transverse beam dynamics in klystron-type buncher.

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