DEGENERATE SOLUTIONS OF THE VLASOV EQUATION

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Abstract

The present report deals with degenerate solutions of the Vlasov equation. By degenerate solution we mean a distribution which has a support of dimension smaller than dimension of the phase space. Well known example is the Kapchinsky-Vladimirsky (KV) distribution, when particles are distributed on the 3-dimensional surface in the 4-dimensional phase space.

We use covariant formulation of the Vlasov equation developed previously [1]. In traditional approach, the Vlasov equation is considered as integro-differential equation with partial derivatives on phase coordinates. For the covariant formulation of the Vlasov equation, we use such tensor object as the Lie derivative. According to the covariant approach, a degenerate solution is described by differential form which degree is equal to the dimension of its support.

Main attention is paid to the KV distribution, which is described by the differential form of the third degree. It is demonstrated that the KV distribution satisfies to the Vlasov equation in covariant formulation.

This work has theoretical as well as practical significance. Presented results can be applied for description and simulation of high-intensity beam.

PHASE SPACE AND PARTICLE DISTRIBUTION DENSITY

Consider a domain D in 4-dimensional space-time and a system of smooth spacelike 3-dimensional surfaces filling the domain D. Introduce a continuous parameterization of that surfaces and system of continuously differentiable one-to-one mappings of those surfaces to some selected surface. Let as call the selected surface a configuration space associated with this foliation of the space-time.

If we specify some reference frame, then we can take the layers of simultaneous events for this reference frame as that surfaces, and the time t a parameter. In this case, the configuration space is the configuration space associated with the reference frame.

When time passes, particles move from one layer to another, but we can examine dynamics of particle ensemble in 3-dimensional configuration space. Let us consider tangent bundle of the configuration space as the phase space. Denote by q a position in the phase space.

If there exists some kind of symmetry, we can pass to a phase space of dimension less than 6. Denote the dimension of the phase space by K.

We shall consider various types of distributions. In the simplest case, consider continuous charged media occupying a domain G_0 in the phase space instead of set of discrete particles. Take a family of subdomains $\{G\}, G \subset G_0$, with smooth boundaries for which their characteristic functions are defined:

$$\chi_G(q) = \begin{cases} 1, & q \in G, \\ 0, & q \notin G. \end{cases}$$

Let us call differential form of $K-$th degree

$$n = n_{1...K}(q)dq^1 \wedge \ldots \wedge dq^K \qquad (1)$$

the particle distribution density in the phase space (or phase density), if for each subdomain G

$$\int_{G_0} \chi_G(q)n(q) = N_G. \qquad (2)$$

Here N_G is the number of particles in G, which in this model may be not integer.

Consider the space of functions $f(q)$ for which $\int_{G_0} f(q)\omega(q)$ exists for any form of $K-$th degree $\omega(q)$ from given class. Let us call such functions integrable and denote by $\mathcal{F}$ their space. For some form $\omega(q)$, define a linear functional on $\mathcal{F}$ by the rule

$$< \omega, f >= \int_{G_0} f(q)\omega(q), \quad f \in \mathcal{F}. \qquad (3)$$

Then definition (2) can be written as

$$< n, \chi_G >= N_G. \qquad (4)$$

Let us consider now the discrete model of point-like particles. In the frames of this model each particle is represented by a point in the phase space. Let us introduce the linear functional $\delta(q)$ on $\mathcal{F}$:

$$< \delta(q), f >= f(q), \quad f \in \mathcal{F}. \qquad (5)$$

The measure $\mu_D =< \delta(q), \chi_D >$ is usually called the Dirac measure. Therefore, let us call the functional (5) the density of the Dirac measure. Let us call a linear combination of functionals (5)

$$< \sum_i \alpha_i \delta(q_{(i)}), f >= \sum_i \alpha_i f(q_{(i)})$$

such that for each subdomain G the equality (4) holds the phase density. It is easy to see that in this case $\alpha_i = 1$, and $q_{(i)}$ are particle positions in the phase space, $i = \overline{1, N}$ where N is the total number of particles:

$$n(q) = \sum_{i=1}^{N} \delta(q_{(i)}). \qquad (6)$$

In this case, the density (6) is described by a scalar function, which is a differential form of 0 degree.

Consider also the model that can be regarded as intermediate case between the model on continuous media and the model of point-like particles. Assume that particles are continuously distributed on some oriented $p-$dimensional surface S in the domain G_0. We shall describe distribution density in this case by a differential form of $p-$th degree defined on the surface. This form depends on orientation of the surface, which is given by a set of $K - p$ vectors.

A form of $p-$th degree $\sigma(q)$ defined on a p-dimensional oriented surface S specifies a functional on $\mathcal{F}$:

$$< \sigma(q), f > = \int_S f(q)\sigma(q).$$

In this case, call such form

$$n(q) = \sigma(q) \qquad (7)$$

that the condition (4) holds the phase density.

COVARIANT FORMULATION OF THE VLASOV EQUATION

As particles moves, their density depends on time. In particular, points and surfaces for the cases (6) and (7) also moves.

According to Vlasov, assume that particle dynamics is determined by an external electromagnetic field and by the self electromagnetic field, which is created by the media being used as the model of a particle ensemble. For continuous models (1), (7), we assume that particle density has sufficiently small components to neglect the collision integral.

The particle dynamics equations define vector field w in the domain D_0 of the phase space. If right hand sides of the dynamics equations are continuously differentiable, then there exist integral lines, unique for each point and each instance of time. Time can be taken as a parameter for integral lines. In the simple case, when the phase space is associated with an inertial frame, the vector field w is defined by particle dynamics equations

$$\frac{dx}{dt} = v, \quad \sum_{i=1}^{3} g_{ik}(\frac{d}{dt}\gamma v)^i = \frac{e}{m}(E_k + \sum_{i=1}^{3} B_{ki}v^i), \quad (8)$$

$k = 1, 2, 3$. Here e and m are charge and mass of a particle, γ is reduced energy (in nonrelativistic case $\gamma = 1$), g_{ik} are components of the metric tensor.

The covariant form of the Vlasov equation is [1]

$$n(t + \delta t, F_{w, \delta t}q) = F_{w, \delta t}n(t, q). \qquad (9)$$

Here $F_{w, \delta\lambda}$ denotes Lie dragging along vector field w by parameter increment δt.

For example, consider continuous model of maximal dimension. Let the phase density is differential form of maximal degree. For simplicity, assume that n is continuously

differentiable on t as a parameter. Then the Vlasov equation can be written in the form

$$\frac{\partial n}{\partial t} = -\mathcal{L}_w n(t, q). \qquad (10)$$

Additionally assume that $\overline{n} = n_{1...K}(t, q)$ is continuously differentiable on phase coordinates, consider nonrelativistic particles, and take Cartesian coordinates. Then the equation (10) means that the single component of the phase density satisfies to the equation

$$\frac{\partial \overline{n}}{\partial t} + \sum_{i=1}^{6} w^i \frac{\partial \overline{n}}{\partial q^i} = 0.$$

CYLINDRICAL BEAM IN LONGITUDINAL MAGNETIC FIELD

Consider nonrelativistic uniformly charged cylindrical beam in uniform longitudinal magnetic field. Assume that all particles have the same longitudinal velocity. Such beam can be described by four-dimensional particle distribution in the phase space of the transverse motion. Integrals of the particles transverse motion are

$$M = r^2(\dot{\varphi} + \omega_0),$$

$$H = \dot{r}^2 + \omega^2 r^2 + M^2/r^2.$$

Here r, φ, z are cylindrical coordinates, $\omega_0 = eB_z/(2m)$, $\omega^2 = \omega_0^2 - e\varrho_0/(m\varepsilon_0)$, e, m are charge and mass of the particles, ϱ_0 is spatial density of the particles inside the beam cross-section, ε_0 is electric constant, B_z is longitudinal component of the magnetic flux density [2, 3]. In this case, magnetic flux is equal to $\Phi = \int B_{r\varphi} dr \wedge d\varphi$ [4]. Hence, $B_z = B_{r\varphi}/r = $ const and $\omega = $ const.

Take φ, phase of a particle trajectory θ, M, and H as coordinates in the phase space. At first, consider the Brillouin flow. In this case $\varrho = m\omega_0^2\varepsilon_0/e$ and $\omega = 0$. Therefore, particles can move only on the surface $M = 0$, $H = 0$. The Vlasov equation in this case can be written in the form

$$\frac{\partial n_{\varphi\theta}}{\partial t} = \frac{d\varphi}{dt}\frac{\partial n_{\varphi\theta}}{\partial \varphi} + \frac{d\theta}{dt}\frac{\partial n_{\varphi\theta}}{\partial \theta}.$$

One can see that $n_{\varphi\theta} = $ const satisfies to the Vlasov equation. It means that the particles are evenly distributed on azimuthal angle φ and on phases of their trajectories θ. As particle trajectories in this case are circles, such distribution means rigid rotation of the beam as a whole around its axis.

Consider well known Kapchinsky-Vladimirsky distribution. Let particles are located on the 3-dimensional surface $H = \omega^2 R^2$, and are evenly distributed on M in segment $M \in [-\omega R^2/2, \omega R^2/2]$, on azimuthal angle φ, and on phases of their trajectories θ [2, 3]: $n_{M\varphi\theta} = $ const. It is easy to see that it is a stationary solution of the Vlasov equation, which in this case has the form

$$\frac{\partial n_{\varphi\theta M}}{\partial t} = \frac{dM}{dt}\frac{\partial n_{\varphi\theta M}}{\partial M} + \frac{d\varphi}{dt}\frac{\partial n_{\varphi\theta M}}{\partial \varphi} + \frac{d\theta}{dt}\frac{\partial n_{\varphi\theta M}}{\partial \theta}.$$

Show that spatial density of particles is uniform in the beam cross-section. Trajectories of the particles can be found from equation of their motion: $\ddot{r} = -\omega^2 r^2 + M^2/r^2$. Integrating it, we obtain

$$r = 2^{-1/2} R \sqrt{1 - 4\mu^2 \cos(2\theta)}, \quad \theta = \omega t + \theta_0$$

where $\mu = M/(\omega R^2)$. Passing to the Cartesian coordinates $x = r \cos \varphi$ and $y = r \sin \varphi$, we get

$$n_{xyM} = \frac{n_{\varphi\theta M}}{D}, \quad D = \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \varphi} - \frac{\partial x}{\partial \varphi}\frac{\partial y}{\partial \theta} = \frac{r^2 \mu \sin 2\theta}{4}.$$

Expressing θ through r one can get

$$\sin 2\theta = \frac{\pm\sqrt{1 - \mu^2 - (1 - 2r^2/R^2)^2}}{\sqrt{1 - \mu^2}}.$$

Then $D = R^2\sqrt{q^2 - q^4 - \mu^2}$ where $q = r/R$. For particles which trajectory passes through point with given r, maximal value of $|M|$ is $M_{\max} = \omega r R \sqrt{1 - r^2/R^2}$. Integrating on M and passing to the variable μ we obtain

$$n_{xy} = \alpha \int_0^{q^2 - q^4} \frac{d\mu}{\sqrt{q^2 - q^4 - \mu^2}} = \text{const}$$

where $\alpha = \text{const}$. Hence, specifying the phase density as mentioned above, we obtain the KV distribution.

KV DISTRIBUTION FOR BEAM IN TRANSVERSE ELECTRIC FIELD

Consider particle distribution in the phase space of the transverse motion. Transverse Cartesian coordinates x, y and corresponding components of velocity $\dot{x}, \dot{y}$ can be taken as coordinates in the phase space. The KV (microcanonical) distribution is the distribution for which particles lie on the surface of the $4-$dimensional ellipsoid, and density in the configuration space is uniform inside corresponding ellipse.

Assume that this ellipsoid is specified by the matrix

$$\begin{pmatrix} B^x & 0 \\ 0 & B^y \end{pmatrix}, \quad B^{x,y} = \begin{pmatrix} (R_{x,y})^{-2} & 0 \\ 0 & (V_{x,y})^{-2} \end{pmatrix}.$$

Let introduce coordinates $\varphi_x, \varphi_y, \theta$, on the surface of the ellipsoid:

$$x = R_x \cos\varphi_x \cos\theta, \quad \dot{x} = V_x \sin\varphi_x \cos\theta,$$

$$y = R_y \cos\varphi_y \sin\theta, \quad \dot{y} = V_y \sin\varphi_y \sin\theta.$$

Take the distribution density on the surface of the ellipsoid in the form

$$n = n_{\varphi_x\varphi_y\theta} \, d\varphi_x \wedge d\varphi_y \wedge d\theta.$$

Find such $n_{\varphi_x\varphi_y\theta}$ that density in the configuration space be uniform inside ellipse $x^2/R_x^2 + y^2/R_y^2 = 1$. Expressing density component in coordinates $x, y, \dot{y}$, we can obtain

$$n_{\varphi_x\varphi_y\theta} = n_{xy\dot{y}} R_x R_y V_x |\sin\varphi_y| |\cos\theta| \sin^2\theta.$$

Integrating on admissible values of $\dot{x}$ we have

$$n_{xy} = \int_{-\dot{x}_{\max}}^{\dot{x}_{\max}} n_{xy\dot{x}} \, d\dot{x} =$$

$$= \int_{-\dot{x}_{\max}}^{\dot{x}_{\max}} \frac{n_{\varphi_x\varphi_y\theta} \, d\dot{x}}{R_x V_x |\sin\theta| |\cos\theta| \sqrt{R_y^2(1 - (\frac{x}{R_x})^2 - (\frac{\dot{x}}{V_x})^2 - y^2}}.$$

Hence, if

$$n_{\varphi_x\varphi_y\theta} = n_0 |\sin\theta| |\cos\theta|, \tag{11}$$

the spatial distribution is uniform.

If beam is propagating in linear transverse electric field: $E_x = kx$, $E_y = -ky$, then it can be shown that θ is the motion integral. So, the density (11) satisfy to the Vlasov equation, which in this case takes form

$$\frac{\partial n_{\varphi_x\varphi_y\theta}}{\partial t} = \frac{d\varphi_x}{dt}\frac{\partial n_{\varphi_x\varphi_y\theta}}{\partial \varphi_x} + \frac{d\varphi_y}{dt}\frac{\partial n_{\varphi_x\varphi_y\theta}}{\partial \varphi_y} + \frac{d\theta}{dt}\frac{\partial n_{\varphi_x\varphi_y\theta}}{\partial \theta}.$$

CONCLUSION

As it is demonstrated, the covariant approach works for description of degenerate solutions of the Vlasov equation. The advantage of this approach is that it can be used when the coordinates are curvilinear. Such approach can be also used for description of the matter distribution in relativity theory.

REFERENCES

[1] O.I.Drivotin, Proc. of the 2011 Int. Particle Accelerators Conf (IPAC'2011) (San-Sebastian, 2011), accelconf.web.cern.ch/accelconf /IPAC2011/papers/wepc114.pdf

[2] O.I.Drivotin, D.A.Ovsyannikov, *Nucl. Instr. Meth. Phys. Res.*, A 558, pp. 112-118, (2006).

[3] O.I.Drivotin, D.A.Ovsyannikov, *Int. J. Modern Phys.*, A 24 (5), pp. 816-842, (2009).

[4] O.I. Drivotin, "Mathematical Basics of the Field Theory", Publ.Company of St.Petersburg State Univ., St.Petersburg, 2010 (in Russ.).