CENTROID AND EMITTANCE OF A KICKED BEAM IN RINGS

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Abstract

We generalize the existing decoherence theory of a kicked beam to linearly uncoupled 4D transverse phase space. Analytical results for the beam centroid motion as well as beam emittance (i.e., 1st and 2nd order beam moments) are given. We formulate the problem in the language of one-turn map and normal form theory in order to emphasize the approximations used and facilitate further generalizations. We will not discuss the chromaticity decoherence factor, which can be found in the references.

1 BEAM PHASE-SPACE DISTRIBUTION AND NORMAL FORM OF ONE-TURN MAP

After a stored beam is kicked, the beam phase space distribution will filament due to nonlinearities. In order to calculate the beam centroid motion and emittance growth, we need to know the phase-space distribution of the kicked bunch. To obtain this time dependent distribution, we employ the basic rule for transforming a continuous distribution f from one set of variables $\vec{u}$ to another $\vec{v} = \mathcal{T}\vec{u}$, which is:

$$f_{\vec{v}}(\vec{v}) = f_{\vec{u}}(\mathcal{T}^{-1}\vec{v})|J_{\mathcal{T}}| \tag{1}$$

where $J_{\mathcal{T}}$ is the Jacobian of the single-valued transformation $\mathcal{T}$. This rule will be used repeatedly although in most cases it will not be mentioned explicitly. Suppose the initial distribution is $\rho_0(\vec{x})$ and the effect of a kick is to translate the equilibrium distribution from $\langle p\rangle = 0$ to $\langle p\rangle = \Delta x'$ without changing positions, the new phase space distribution obviously is $\rho(x,p) = \rho_0(x, p-\Delta x')$.

The n-th turn phase space distribution $\rho_n(\vec{x})$ depends on the one-turn map $\mathcal{M}$ of the ring, i.e., $\rho_n(\vec{x}) = \rho(\mathcal{M}^{-n}\vec{x})$, where we have used the fact that the Jacobian of the one-turn map equals to 1 due to symplecticity. In general, the map $\mathcal{M}$ could be extremely complicated and numerical tracking is the only way to proceed, even though such tracking is very time-consuming. However, for a normal operating ring, the one-turn map usually has very weak nonlinearity and the tune is placed away from significant resonances. Therefore, it is safe to assume that there is an effective normal form

$$\mathcal{N} = \mathcal{A}^{-1} \circ \mathcal{M} \circ \mathcal{A} \tag{2}$$

for the map of a normal operational storage ring. Now the phase space distribution becomes:

$$\rho_n(\vec{x}) = \rho(\mathcal{A} \circ \mathcal{N}^{-n} \circ \mathcal{A}^{-1}\vec{x}) \tag{3}$$

Since the normal form represents a simple rotation in the normalized phase space $\mathcal{A}^{-1}\vec{x}$, it is much easier to work with this normalized phase space. Using action-angle variables $\vec{J} \equiv \{J_x, \phi_x, J_y, \phi_y\}$, the n-th turn phase space distribution reads:

$$\begin{aligned}\rho_{\vec{J}}(\vec{J}) &= \rho_{\vec{x}}(\mathcal{A} \circ \mathcal{N}^{-n}\vec{J}) \\ &= \rho_{\vec{x}}(\mathcal{A}\{J_x, \phi_x - 2\pi\nu_x n, J_y, \phi_y - 2\pi\nu_y n\})\end{aligned} \tag{4}$$

In Eq. (4) we only assumed that the single particle motion can be canonically transformed (via a nonlinear map $\mathcal{A}$) into a circular phase space motion with amplitude dependent tunes ν_x and ν_y. To further simplify the calculation, we adopt the following major approximations:

- Take only the leading terms in the normal form $\mathcal{N}$, i.e., consider only the linear tune-shift-with-amplitude terms.
- Drop the nonlinear part of $\mathcal{A}$, i.e., just use the linear Courant-Synder normalization and the well-known action-angle variables $\{J, \phi\}$.

Note that these approximations, especially the last one when nonlinear part of $\mathcal{A}$ is large, may limit the usefulness of the result. In the following, we limit our discussion to the case of linearly uncoupled 4D transverse phase space.

Let us start with a 2D well-damped beam. Due to radiation damping and quantum excitation, the well-known equilibrium distribution is:

$$\rho_0(x,p) = \frac{1}{2\pi\epsilon} e^{-\frac{1}{2\epsilon}(\gamma x^2 + 2\alpha x p + \beta p^2)} \tag{5}$$

where α, β, γ, and ϵ are the Twiss parameters and emittance. Using Eq. (4) and the Courant-Synder normalization matrix $A = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\alpha/\sqrt{\beta} & 1/\sqrt{\beta} \end{pmatrix}$, the action-angle phase space distribution right after a kick becomes:

$$\begin{aligned}&\rho(J,\phi) \\ &\simeq \rho_0\left(\sqrt{2J\beta}\cos\phi, -\sqrt{2J/\beta}\,(\sin\phi + \alpha\cos\phi) - \Delta x'\right) \\ &\simeq \frac{1}{2\pi\epsilon} e^{-\frac{1}{2}\left[2J/\epsilon + 2\sqrt{2J/\epsilon}\,k\,\sin\phi + k^2\right]}\end{aligned} \tag{6}$$

where $k \equiv \sqrt{2\Delta J/\epsilon} = \frac{\beta}{\sigma}\Delta x'$ is a dimensionless quantity that measures the effective kick ΔJ on the action. For the n-th turn, one simply shifts the phase ϕ by $-2\pi(\nu_0 + aJ)n$, where ν_0 and a are the linear tune and tune-shift-with-amplitude.

Now we consider the 4D case with negligible linear coupling between the x and y degree of freedom. The total phase space distribution is a direct product of the distributions in the two 2D subspace. However, there is an extra tune-shift term contributing to the decoherence process which must be taken into account because, in the 4D case, the tunes reads:

$$\begin{aligned}\nu_x &= \nu_x^0 + a_x J_x + a_{xy} J_y + \cdots \\ \nu_y &= \nu_y^0 + a_{xy} J_x + a_y J_y + \cdots \end{aligned} \quad (7)$$

where a_x, a_{xy}, and a_y are the tune-shift-with-amplitude coefficients. In summary, the 4D phase space distribution of a beam kicked to $\{0, \Delta x', 0, \Delta y'\}$ is:

$$\begin{aligned} &\rho(J_x, \phi_x, J_y, \phi_y) \\ \simeq\ & \frac{1}{2\pi\epsilon_x} e^{-\frac{1}{2}\left(2J_x/\epsilon_x + 2\sqrt{2J_x/\epsilon_x}\, k_x \sin\Phi_x + k_x^2\right)} \cdot \\ & \frac{1}{2\pi\epsilon_y} e^{-\frac{1}{2}\left(2J_y/\epsilon_y + 2\sqrt{2J_y/\epsilon_y}\, k_y \sin\Phi_y + k_y^2\right)} \end{aligned} \quad (8)$$

where all the symbols are subscribed with x and y to indicate the horizontal and vertical planes. $\Phi_x \equiv \phi_x - 2\pi(\nu_x^0 + a_x J_x + a_{xy} J_y)n$ and similar shorthand holds for Φ_y.

2 BEAM CENTROID

Using the phase space distribution of Eq. (8), the centroid motion is simply given by:

$$\begin{aligned}\langle x \rangle &= \langle \mathcal{A}\{J_x, \phi_x, J_y, \phi_y\}\rangle|_{x \text{ component}} \\ &\simeq \langle \sqrt{2J_x\beta_x} \cos\phi_x \rangle \end{aligned} \quad (9)$$

$$\begin{aligned}\langle p_x \rangle &= \langle \mathcal{A}\{J_x, \phi_x, J_y, \phi_y\}\rangle|_{p_x \text{ component}} \\ &\simeq -\langle \sqrt{2J_x\gamma_x} \cos(\phi_x - \frac{\pi}{2} + \arctan\alpha_x) \rangle \end{aligned}$$

where $\langle \cdots \rangle$ means average over phase space distribution. These first moments rely on the integral:

$$\begin{aligned} \langle \sqrt{2J_x} \cos(\phi_x + \phi_0) \rangle &= \int_0^\infty d\,J_x d\,J_y \int_0^{2\pi} d\phi_x d\phi_y \\ & \quad \rho(J_x, \phi_x, J_y, \phi_y) \sqrt{2J_x} \cos(\phi_x + \phi_0) \\ &= k_x \sqrt{\epsilon_x}\, \bar{A} \sin(2\pi\nu_x^0 n + \Delta\bar{\phi} + \phi_0) \end{aligned} \quad (10)$$

where

$$\begin{aligned} \bar{A} &\equiv \frac{1}{1+\theta_x^2} e^{-\frac{k_x^2}{2}\frac{\theta_x^2}{1+\theta_x^2}} \cdot \frac{1}{\sqrt{1+\theta_{xy}^2}} e^{-\frac{k_y^2}{2}\frac{\theta_{xy}^2}{1+\theta_{xy}^2}} \\ \Delta\bar{\phi} &\equiv \frac{k_x^2}{2}\frac{\theta_x}{1+\theta_x^2} + 2\arctan\theta_x + \\ & \quad \frac{k_y^2}{2}\frac{\theta_{xy}}{1+\theta_{xy}^2} + \arctan\theta_{xy} \end{aligned} \quad (11)$$

and $\theta_x \equiv 2\pi a_x \epsilon_x n$, $\theta_{xy} \equiv 2\pi a_{xy} \epsilon_y n$.

In summary, we obtained the centroid motion of a kicked beam in uncoupled 4D transverse phase space as:

$$\begin{aligned} \langle x \rangle &\simeq k_x \sigma_x\, \bar{A} \sin(2\pi\nu_x^0 n + \Delta\bar{\phi}) \\ \langle p_x \rangle &\simeq k_x \sigma_{x'} \bar{A} \cos(2\pi\nu_x^0 n + \Delta\bar{\phi} + \arctan\alpha_x) \end{aligned} \quad (12)$$

Obviously, similar results hold for $\langle y \rangle$ and $\langle p_y \rangle$. Comparing this to the 2D result in Ref. [1, 5], there are extra contributions to the decoherence factor $\bar{A}$ and phase shift $\Delta\bar{\phi}$ due to the new transverse degree of freedom and nonlinear coupling, even when the beam was kicked in one plane only. These modifications are particularly important when considering beam decoherence in the vertical plane, because the horizontal emittance is usually much larger than the vertical one. The effect of non-zero α is also included in Eq. (12) explicitly. For kicks other than the one we have used, the phase and amplitude need to be adjusted to satisfy the initial condition as for a single particle.

3 BEAM SIZE AND EMITTANCE

For many applications, it would be important to know the rms emittance and other second order beam moments as the beam decoheres. The calculation of these moments rely on the following two integrals:

$$\begin{aligned} \langle J_x \rangle &= \frac{1}{\epsilon_x\epsilon_y} e^{-\frac{k_x^2+k_y^2}{2}} \int_0^\infty dJ_x J_x e^{-\frac{J_x}{\epsilon_x}} I_0(\sqrt{\frac{2J_x}{\epsilon_x}}\, k_x) \\ & \qquad \int_0^\infty dJ_y e^{-\frac{J_y}{\epsilon_y}} I_0(\sqrt{\frac{2J_y}{\epsilon_y}}\, k_y) \\ &= \epsilon_x(1 + \frac{1}{2}k_x^2) \end{aligned} \quad (13)$$

and

$$\begin{aligned} & \langle J_x \cos(2\phi_x + \phi_0) \rangle \\ &= -\frac{1}{\epsilon_x\epsilon_y}\, e^{-\frac{k_x^2+k_y^2}{2}} \int_0^\infty dJ_y\, e^{-\frac{J_y}{\epsilon_y}}\, I_0(\sqrt{\frac{2J_y}{\epsilon_y}}\, k_y) \\ & \quad \int_0^\infty dJ_x\, J_x\, e^{-\frac{J_x}{\epsilon_x}}\, I_2(\sqrt{\frac{2J_x}{\epsilon_x}}\, k_x)\, \cos[4\pi\nu_x n + \phi_0] \\ &= -\epsilon_x \frac{k_x^2}{2} \tilde{A} \cos(4\pi\nu_x^0 n + \Delta\tilde{\phi} + \phi_0) \end{aligned} \quad (14)$$

where I_0 and I_2 are modified Bessel functions,

$$\begin{aligned} \tilde{A} &\equiv \frac{1}{[1+(2\theta_x)^2]^{3/2}}\, e^{-\frac{k_x^2}{2}\frac{(2\theta_x)^2}{1+(2\theta_x)^2}} \cdot \\ & \quad \frac{1}{\sqrt{1+(2\theta_{xy})^2}}\, e^{-\frac{k_y^2}{2}\frac{(2\theta_{xy})^2}{1+(2\theta_{xy})^2}} \\ \Delta\tilde{\phi} &\equiv \frac{k_x^2}{2}\frac{2\theta_x}{1+(2\theta_x)^2} + 3\arctan 2\theta_x + \\ & \quad \frac{k_y^2}{2}\frac{2\theta_{xy}}{1+(2\theta_{xy})^2} + \arctan 2\theta_{xy} \end{aligned} \quad (15)$$

Using Eqs. (13)&(14) we obtain:

$$\begin{aligned}
&\langle x^2 \rangle \qquad (16)\\
&= \beta_x \left(\langle J_x \rangle + \langle J_x \cos 2\phi_x \rangle \right)\\
&= \sigma_x^2 \left[1 + \frac{1}{2} k_x^2 - \frac{1}{2} k_x^2 \tilde{A} \cos(4\pi\nu_x^0 n + \Delta\tilde{\phi}) \right]
\end{aligned}$$

$$\begin{aligned}
&\langle p_x^2 \rangle \qquad (17)\\
&= \gamma_x \left(\langle J_x \rangle - \langle J_x \cos(2\phi_x + 2\arctan\alpha_x) \rangle \right)\\
&= \sigma_{p_x}^2 \left[1 + \frac{1}{2} k_x^2 + \right.\\
&\qquad \left. \frac{1}{2} k_x^2 \tilde{A} \cos(4\pi\nu_x^0 n + \Delta\tilde{\phi} + 2\arctan\alpha_x) \right]
\end{aligned}$$

$$\begin{aligned}
&\langle x\, p_x \rangle \qquad (18)\\
&= -\alpha_x \langle J_x \rangle - \sqrt{\beta_x \gamma_x}\, \langle J_x \cos(2\phi - \frac{\pi}{2} + \arctan\alpha) \rangle\\
&= -\alpha_x \epsilon_x (1 + \frac{k_x^2}{2}) \; + \\
&\quad \frac{\epsilon_x}{2} \sqrt{\beta_x \gamma_x} k_x^2 \tilde{A} \sin(4\pi\nu_x^0 n + \Delta\tilde{\phi} + \arctan\alpha)
\end{aligned}$$

From the first and second order moments, it is easy to get the rms values, such as

$$\begin{aligned}
&\sigma_x^{\mathrm{rms}}(n)\\
&\equiv \sqrt{\langle x^2 \rangle - \langle x \rangle^2}\\
&= \sigma_x \left\{ 1 + \frac{k_x^2}{2}(1 - \bar{A}^2) \; + \frac{k_x^2}{2} [\bar{A}^2 \cos(4\pi\nu_x^0 n + 2\Delta\bar{\phi}) \right.\\
&\qquad \left. - \tilde{A} \cos(4\pi\nu_x^0 n + \Delta\tilde{\phi})] \right\}^{-\frac{1}{2}} \qquad (19)
\end{aligned}$$

The rms emittance becomes:

$$\begin{aligned}
&\epsilon_x^{\mathrm{rms}}(n)\\
&\equiv \sqrt{\left\langle (x - \bar{x})^2 \right\rangle \left\langle (p_x - \bar{p}_x)^2 \right\rangle - \langle (x - \bar{x})\, (p_x - \bar{p}_x) \rangle^2}\\
&= \epsilon_x \left\{ 1 + k_x^2 (1 - \bar{A}^2) + \frac{k_x^4}{4} \left[1 - 2\bar{A}^2 - \tilde{A}^2 + \right. \right.\\
&\qquad \left. \left. 2\bar{A}^2 \tilde{A} \cos(2\Delta\bar{\phi} - \Delta\tilde{\phi}) \right] \right\}^{-\frac{1}{2}} \qquad (20)\\
&\simeq \epsilon_x \sqrt{1 + 2k_x^2 \theta_x^2 + k_x^2 (1 + k_y^2) \theta_{xy}^2 + O(\theta^4)}
\end{aligned}$$

where $k_x \equiv \sqrt{\frac{2\Delta J_x}{\epsilon_x}}$ and $k_y \equiv \sqrt{\frac{2\Delta J_y}{\epsilon_y}}$. Since both $\bar{A}$ and $\tilde{A}$ decay to 0 as the time increases, it is obvious that $\epsilon_x^{\mathrm{rms}} \to \epsilon_x (1 + \frac{1}{2} k_x^2) = \langle J_x \rangle$. Note that this is the limit without radiation damping. The beam will finally be damped into the equilibrium state of emittance ϵ_x. The rms emittance of Eq. (20) has a very interesting property: it does not depend on the linear tunes ν_x^0 and ν_y^0. The Taylor expended expression is good for a relatively small number of turns after the kick, where $\theta \propto n$ is small. By switching the subscripts, all expressions apply to the vertical plane as well.

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5 REFERENCES

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