APPROXIMATE ANALYTICAL DESCRIPTION OF THE UNDERDENSE PLASMA LENS

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Abstract

The perturbative approach for describing the underdense plasma–ultrarelativistic electron bunch system is developed,using the ratio $\frac{n_0}{n_b}$ as a small parameter (n_b-bunch,n_0-plasma electron densities). It is shown that in the zero and first approximations for the ultrarelativistic bunch case the role of positive ion column is negligible ($\sim\ O(\gamma^{-2})$, where γ_0 is the Lorentz factor of the bunch) for the arbitrary bunch length. Focusing of the electron bunch emerged in the first approximation of the perturbative procedure as a result of the plasma electrons redistribution.Focusing gradient and strength for ultrarelativistic,flat,uniform and short bunch are obtained and compared with the previous results, which differs drasticaly from the obtained ones. The plasma chamber conducting walls effect on focusing force is considered by the method of images, adopting the diffuse scattering of plasma electrons on the walls. Effect is small, when the plasma chamber width is much larger than that of bunch.

1 INTRODUCTION

Plasma focusing devices,being compact,simple and effective elements are promissing for obtaining electron (positron) beams of very small spot size, requested for future high energy linear colliders.Theoretical predictions and investigations of the plasma lenses in overdense ($n_b < \frac{1}{2}n_0 \equiv n_p$) and underdense ($n_b > \frac{1}{2}n_0, n_b$-bunch electron density,$n_0$-cold neitral plasma electron density),performed last decade and were supplemented by experimental tests,carried out at ANL, Tokio University-KEK and UCLA for overdense plasma lens regime.

The theoretical treatment of the overdense plasma lens,performed in linear approximation,using ratio $\frac{n_b}{n_0}$ as a small parameter,is more or less complete,at least in the frame of the cold plasma and rigid electron bunch approximations.

The existing theoretical approaches to underdense plasma lens are phenomenological by nature.

It seems,that more detailed description of the underdense plasma lens is needed,in particular, taking into account the program for the future experimental investigations, which include underdense regime too.

Using the ratio $\frac{n_0}{n_b} \ll 1$ as a small parameter,and developing the subsequent perturbative approach for the description of the underdense plasma lens,it is possible to achieve,at least the same level of the understanding of the focusing phenomenon in underdense regime as that of overdense plasma lens case.

2 PERTURBATIVE APPROACH FOR UNDERDENSE PLASMA LENS

Consider the flat electron bunch with the vertical dimension $2b$,which is assumed much smaller than horizontal dimension $2a$; longitudinal dimensions,which are arbitrary,are $2d$.The bunches of such a geometry are suitable for future high energy linear electron-positron colliders.

Bunch electrons uniform density is n_b and bunch is considered as ultrarelativistic and rigid one.Plasma electrons density is $n_0 \ll n_b$,plasma is neutral,cold and with the immobile ions.

The geometry of the bunch,which moves in lab system through the plasma with constant velocity v_0,allows to consider the electric and magnetic fields components as follows:

$E_x = 0, E_y, E_z \neq 0; B_x \neq 0, B_y = B_z = 0$

All physical quantities of the problem are considered as a functions of the arguments y and $\tilde{z} = z - v_0 t$ only (steady state regime).Introduce the dimensionless arguments $y', \tilde{z'} = k_b y, k_b \tilde{z}; t' = \omega_b t, \omega_b^2 = \frac{4\pi e^2 n_b}{m}, k_b = \frac{\omega_b}{c}$,and dimensionless variables $E'_{y,z} = (\frac{\omega_b mc}{e})^{-1} E_{y,z}; B' = (\frac{\omega_b mc}{e})^{-1} B_x$;

Define the generalized plasma electron velocities as

$V_y = \frac{v_{ey}}{v_0 - v_{ez}} = \frac{\beta_{ey}}{\beta - \beta_{ez}}, \beta = \frac{v_0}{c}, \beta_{ez,y} = \frac{v_{e,zy}}{c}$, $V_z = \frac{v_{ez}}{v_0 - v_{ez}} = \frac{\beta_{ez}}{\beta - \beta_{ez}}$

and generalized plasma electron density as

$N = \frac{n_e}{n_b}(1 - \frac{v_{ez}}{v_0}) = \frac{n_e}{n_b}(1 - \frac{\beta_{ez}}{\beta}), \frac{n_e}{n_b} = N(1 + V_z)$,
(BTFCh transformations [1]).

Adopting the condition $\frac{n_o}{n_b} \ll 1$,decompose the quantities in question in the following series:

$$\begin{aligned} N &= \epsilon N_1 + \epsilon^2 N_2 + \ldots \\ V_z &= V_{z0} + \epsilon V_{z1} + \epsilon^2 V_{z2} + \ldots \\ V_y &= V_{y0} + \epsilon V_{y1} + \epsilon^2 V_{y2} + \ldots \\ E_{y,z} &= E_{y,z0} + \epsilon E_{y,z1} + \epsilon^2 E_{y,z2} + \ldots \\ B &= B_0 + \epsilon B_1 + \epsilon^2 B_2 + \ldots \end{aligned} \tag{1}$$

where $\epsilon = \frac{n_0}{n_b}$ (in what follows ϵ^n is included in subsequent quantities N_n, V_{nz} and so on)

In the zero order approximation from

$$\begin{aligned} B_0 &= -\beta E_{y0}, \\ E_{z0} &= -\frac{1}{\gamma^2}\frac{\partial \phi_0}{\partial z}, E_{y0} = -\frac{\partial \phi_0}{\partial y} \\ \frac{1}{\gamma^2}\frac{\partial^2 \phi_0}{\partial z^2} &+ \frac{\partial^2 \phi_0}{\partial y^2} = \begin{cases} 1, & |z| \leq d, |y| \leq b, \\ 0, & |z| > d, |y| > b; \end{cases} \end{aligned} \tag{2}$$

In the first order approximation continuity equations have a form:

$$-\frac{\partial N_1}{\partial z} + \frac{\partial N_1 V_{y0}}{\partial y} = 0 \qquad (3)$$

and Coulomb law:

$$\frac{\partial^2 E_{z1}}{\partial z^2} + \gamma^2 \frac{\partial^2 E_{z1}}{\partial y^2} = \frac{\partial}{\partial z}\left(\frac{n_0}{n_b} - \gamma^2 N_1 - N_1 V_{z0}\right); \quad (4)$$

The focusing force is

$$\begin{aligned} f_y &= -e(E_y + \beta B) = f_{y0} + f_{y1} + ..., \qquad (5)\\ f_{y0} &= -e(E_{y0} + \beta B_0) = -e\gamma^{-2} E_{y0}, \\ f_{y1} &= -e(E_{y1} + \beta B_1) \equiv -eW_{y1}. \end{aligned}$$

In the first approximation

$$\frac{\partial W_{y1}}{\partial \tilde{z}} = \frac{\partial E_{z1}}{\partial y}, \qquad (6)$$

which is an analog of the Panoffsky-Wenzel relation;remember that in (6) $\tilde{z} = z - v_0 t$. Differentating (4) over y and then integrating it over $\tilde{z}$, taking into account relation (6) and that

$\frac{\partial}{\partial y}\left(\frac{n_0}{n_b} - \gamma^2 N_1 - N_1 V_{z0}\right) \rightarrow 0$ when $y \rightarrow \pm\infty$,we have

$$\frac{\partial^2 W_{y1}}{\partial \bar{z}^2} + \frac{\partial^2 W_{y1}}{\partial y^2} = \frac{1}{\gamma^2}\frac{\partial}{\partial y}\left(\frac{n_0}{n_b} - \gamma^2 N_1 - N_1 V_{z0}\right) \quad (7)$$

where $\bar{z} \equiv \gamma\tilde{z}$.

In what follows, consider the ultrarelativistic bunches,when $\gamma \gg 1$; then $f_{y0} \sim O(\gamma^{-2})$ and in right hand side of (7) it is possible to leave only the term $-\frac{\partial N_1}{\partial y}$. So for the sought quantity W_{y1} we need N_1 from eq. (3).Entered in (3) V_{y0} must be found from eqs. of motion for the fields in zero approximation.It is evident,from eq. (7),that the role of the noncompensated positive ions (the term on the right hand side of eq. (7), proportional to $\frac{n_0}{n_b} - N_1(1 + V_{z0})$),is negligible in ultrarelativistic limit ($\sim O(\gamma^{-2})$).

3 ZERO ORDER APPROXIMATION

In the zero order approximation it is necessary to find from (**??**) E_{0y}, E_{0z} for the flat bunch moving in vacuum.

The approximate results for $\gamma \gg 1$ are the following:

$$E_{y0} \approx -y\left(1 - \frac{2bd}{\pi\gamma(d^2 - z^2)}\right) = -y + O(\gamma^{-1}) \qquad (8)$$
$$-b \leq y \leq b, -d < z < d;$$

$$E_{y0} \approx -b\left(1 - \frac{2dy}{\pi\gamma(d^2 - z^2)}\right) = -b + O(\gamma^{-1}) \qquad (9)$$
$$\gamma d \gg y \geq b, -d < z < d;$$

when $|z| \ll d$ the expression (9) is valid for $\gamma d \leq y \leq b$.

$$E_{y0} \approx -\frac{2\gamma db}{\pi y} \qquad (10)$$
$$y > \gamma d \gg b, -d \leq z \leq d;$$

$$E_{y0} \approx -\frac{2dby}{\pi\gamma z^2} \qquad (11)$$
$$z \gg d, \gamma d \gg b, -b \leq y \leq b;$$

The next problem in zero approximation is the definition of the generalized velocities V_{y0}, V_{z0}.When $\gamma \gg 1$,and bunch length is short enough $d' \ll 1, (d' = k_b d)$ it is possible to drop out in eqs. of motion the terms proportional to $|V_{z0}|^2 \ll |V_{z0}|$ and to $|V_{y0}|^2 \ll |V_{y0}|$. Taking into account that in the considered case $|E_{y0}| \gg |E_{z0}|$ the approximate solutions for the boundary condition $V_{y0} = 0, V_{z0} = 0$, when $z = d$ are:

$$\begin{aligned} V_{y0} &= \sinh\left(|E_{y0}|(d - z)\right) \simeq |E_{y0}|(d - z), \qquad (12)\\ V_{y0}(-y) &= -V_{y0}(y), \\ V_{z0} &= \cosh\left(|E_{y0}|(d - z)\right) - 1 \simeq \frac{1}{2}|E_{y0}|^2 (d - z)^2, \end{aligned}$$

where E_{y0} is given by (8-10)and $V_{y0} \rightarrow 0, V_{z0} \rightarrow 0$ when $y \rightarrow \pm\infty$. For the short ($d \ll 1$) bunches from (12) follows that $|V_{z0}| \ll |V_{y0}|$.

4 FOCUSING FORCE IN THE FIRST ORDER APPROXIMATION

Using eq. (3) in the first approximation and eq. (12) for V_{y0} it is possible to find $N_1(y, z) = N_1(-y, z)$ by the method of characteristics. The plasma electrons redistribution function $N_1(y, z)$ has a crest at distances $y_{cr} \approx \frac{2\gamma d}{\pi} \gg b$, conditioned by bunch electric field E_{0y}, exponentialy rising with y at $|y| < y_{cr}$ and has a power fall at $|y| \gg y_{cr}, |y| \rightarrow \infty$

For $\gamma \gg 1$ in eq. (7) we get an approximate solution

$$W_{y1} = -\frac{1}{2\pi}\int d\bar{z}' \int dy' \frac{\partial N_1(y', \bar{z}')}{\partial y'} \qquad (13)$$
$$\ln\left[(y - y')^2 + (\bar{z} - \bar{z}')^2\right]^{1/2}.$$

The domain of the integration in (13) is defined by the condition $\frac{\partial N_1(y,z)}{\partial y} \neq 0$,i.e. it is $-\infty < y < +\infty, -\infty < z \leq d$,because for $z > d, N_1 = \frac{n_0}{n_b}$. For $z < -d$ let us assume $N_1(z, y) = N_1(-d, y)$ up to some $-z_0 < -d$ where considered steady state regime changes to (nonlinear) wake wave (and later on the uniform distribution of the plasma electrons with $N_1 = \frac{n_0}{n_b}$).

For the estimates,it is possible to use approximate expression, which follows from (13):

$$W_{y1} \approx \frac{8}{\pi} y d^2 \alpha(z) \frac{n_0}{n_b}, \quad \alpha(z) \sim 1 \qquad (14)$$

Focusing force (14) is the result of the long-range redistribution ($y \sim y_{cr} \approx \frac{2}{\pi}\gamma d$) of plasma electrons caused by ultrarelativistic bunch electric field E_{0y} (8,9,10), which is practicaly non-screened in the case of underdense plasma. Hence the question is arised on the effect of the walls of the plasma chamber.

The problem can be solved by the method of images in the model of a plasma chamber as a flat plasma layer inside the two conducting planes with distance 2g in between. Adopting diffuse scattering of plasma electrons on the planes, the results of such a consideration can be formulated as following: $g \gg \gamma d \gg b$ the effect of plasma walls on focusing force (14) is negligible; when $\gamma d \gg g \gg b$ the decrease of quantitative factor in (14) takes place.

5 DISCUSSION

The expressions for the, focusing gradient and stregth are subsequently

$$G = \frac{W_{1y}}{y} = \frac{128\alpha(z)er_e}{\pi}(n_b n_0)d^2, \qquad (15)$$
$$K = \frac{eG}{\gamma mc^2} = \frac{128\alpha(z)r_e^2 d^2}{\gamma\pi}(n_b n_0),$$

The expressions (15) differs from the previous results,which for the flat, electron bunch and underdense plasma are ($r_e = e^2/mc^2$ is the electron classical radius):

$$G = \frac{W_{1y}}{y} = 4\pi n_0 e, K = \frac{4\pi r_e n_0}{\gamma} \qquad (16)$$

These expressions are based on the understanding the focusing phenomenon as caused by the positive charge of the ion column,which exists inside and behind the moving electron bunch, because the considered bunch blows out the plasma electrons.

As it follows from the presented consideration (7), the net focusing effect of the ion column,caused by noncompensated positive charge $\frac{n_0}{n_b} - N_1(1+V_{z0})$, is proportional to γ^{-2} and is negligible in our approximation $\gamma \gg 1$,compared to focusing effect of the plasma electrons redistribution,described by eqs. (13) and (14).The presence of the factor γ^{-2} in the part of the net (electric plus magnetic) focusing force,which caused by ions,is a result of magnetic compesation of the subsequent focusing electric field in lab frame,where electron bunch moves with the velocity v_0 and unperturbed parts of plasma are at rest. Compensating magnetic field is caused by noncompensated positive ions "current", which propogate with the velocity v_0,equal to the velocity of the electron bunch and at $\gamma \gg 1$ cancels the focusing action of the subsequent electric field.The physical reason for that - the exposed positive ions column effectively "moves" in plasma along with the moving electron bunch with the velocity of it.When bunch blows out the plasma electrons,it uncower new forward parts of the partialy noncompensated ion column and this "motion" of the revealed positive ions charge,i.e. subsequent current,generated compensating magnetic field.

The physical interpretation can be based also directly on the displacement current,entered in the Maxwell eq. through the term $\beta\frac{\partial E_{z1}}{\partial z}$.Ions ahead of the bunch are completely compensated by plasma electrons,inside the bunch ions charge is only partially compensated,so exists the change of electric field component with $\tilde{z} = z - v_0 t$, which generates subsequent compensating magnetic field.

All above mentioned general physical arguments are valid for the relativistic bunches with arbitrary length and charge distribution.Assumptions adopted in the present work (short enough relativistic bunch with the uniform charge distribution) allows us to perform analytical calculations (14).

Unfortunately the condition $k_b d \ll 1$ ($d' \ll 1$,short bunches) does not permit to use expression (15) for focusing strength for the flat beams of the FFTB. In that case $n_b = 7,7 \cdot 10^{18} cm^{-3}$ and $2,8 \cdot 10^{18} cm^{-3}$ and $k_b = 5,2 \cdot 10^5 cm^{-1}$ and $k_b = 3,1 \cdot 10^5 cm^{-1}$ consequently, $d = 2,3 \cdot 10^{-2} cm$ and $k_b d = 1,2 \cdot 10^4 \gg 1$ and $0,7 \cdot 10^4 \gg 1$.

Using the short bunches,nevertheless it is possible to obtain sufficiently strong focusing gradients.For example, let $k_b d = 0,1$.Then $d = 10^{-5}\left(\frac{3\cdot 10^{19}}{n_b}\right)^{1/2}$, $G = 15 \cdot 10^{-11} n_0$ and if $n_0 \approx 10^{17} cm^{-3}$ as in FFTB case,$G \approx 15\frac{MG}{cm}$.In order to fullfil the condition $\frac{n_0}{n_b} \ll 1$, n_b must be e.g.,equal $10^{18} cm^{-3}$,then $d = 5 \cdot 10^{-5} cm$.

Probably the medium and long bunch cases can be treated by computational methods only.Presented analytical approximate description of the short bunch case can be used then as a physical guide and programmes testing example.

It will be interesting to test experimentaly the predictions for focusing gradient and strenght,particularly,the dependense of these quantities on n_b and d.

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7 REFERENCES

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