GLOBAL AND LOCAL HORIZONTAL-VERTICAL DECOUPLING

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Abstract

In this paper we investigate systematically the global and local effects of horizontal-vertical couplings on the beam and the measurement and control of these couplings. Piecemeal studies have been made on these subjects by different authors [1, 2, 3]. For completeness, their results are integrated here wherever appropriate. This brings the understanding and execution of the coupling correction to the same degree of completeness as that of the closed orbit correction.

1 GENERAL PROPERTIES OF SYMPLECTIC 4 × 4 MATRICES

A 4 × 4 matrix is conveniently written as

$$T = \left(\begin{array}{c|c} M & m \\ \hline n & N \end{array}\right). \qquad (1)$$

The cross separates the matrix into four 2 × 2 matrices. The symplectic conjugate of T is

$$T^{+} \equiv \left(\begin{array}{c|c} S & 0 \\ \hline 0 & S \end{array}\right)\left(\begin{array}{c|c} \tilde{M} & \tilde{n} \\ \hline \tilde{m} & \tilde{N} \end{array}\right)\left(\begin{array}{c|c} \tilde{S} & 0 \\ \hline 0 & \tilde{S} \end{array}\right) = \left(\begin{array}{c|c} M^{+} & n^{+} \\ \hline m^{+} & N^{+} \end{array}\right) \qquad (2)$$

where

$$S \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \text{unit symplectic matrix.}$$

T is symplectic if $T^{+}T = 1$, namely $T^{+} = T^{-1}$ = inverse.

A symplectic matrix can be parameterized as:

$$T = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & N \end{array}\right)\left(\begin{array}{c|c} 1 & F \\ \hline -F^{+} & 1 \end{array}\right) = \left(\begin{array}{c|c} M & MF \\ \hline -NF^{+} & N \end{array}\right) \qquad (3)$$

or as

$$T \equiv RPR^{+}$$

$$\text{with } R = \left(\begin{array}{c|c} \cos\psi & D\sin\psi \\ \hline -D^{+}\sin\psi & \cos\psi \end{array}\right),\ P = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right). \qquad (4)$$

In the first form, Eq. (3), the 2 × 2 matrices M, N, and F are not symplectic, but the number of free parameters are reduced from 12 to 10 by the relations det M = det N = $(1 + \det F)^{-1}$. In the second form, Eq. (4), A, B, and D are all symplectic 2 × 2 matrices. Together with ψ, this form contains again the requisite ten parameters. Relations among the various forms are:

$$\begin{cases} A = M - Dn\tan\psi & D = h/\sqrt{\det h} \\ B = N + D^{+}m\tan\psi & \tan 2\psi = \dfrac{\sqrt{\det h}}{\frac{1}{2}\mathrm{Tr}(N-M)}, \end{cases} \qquad (5)$$

where the fundamental coupling matrix h is

$$h \equiv m + n^{+} = MF - FN^{+}.$$

2 ONE-TURN TRANSFER MATRIX

We write the coordinate 4-vector of the transverse phase space as $\begin{pmatrix} X \\ Y \end{pmatrix}$ with $X \equiv \begin{pmatrix} x \\ x' \end{pmatrix}_o$ and $Y \equiv \begin{pmatrix} y \\ y' \end{pmatrix}$, and denote by T the 4 × 4 symplectic one-turn transfer matrix. Equation (4) shows that the decoupled principal coordinate 4-vector is

$$\begin{pmatrix} U \\ V \end{pmatrix} \equiv R^{+}\begin{pmatrix} X \\ Y \end{pmatrix} \quad \text{with } U = \begin{pmatrix} u \\ u' \end{pmatrix},\ V \equiv \begin{pmatrix} v \\ v' \end{pmatrix} \qquad (6)$$

and is transferred by the decoupled matrix P.

The principal betatron tunes (eigen-tunes) are given as usual by

$$\cos 2\pi\nu_u = \frac{1}{2}\mathrm{Tr}A, \quad \cos 2\pi\nu_v = \frac{1}{2}\mathrm{Tr}B. \qquad (7)$$

From Eq. (5) we can derive the relationship

$$(\cos 2\pi\nu_v - \cos 2\pi\nu_u)^2 = \left[\frac{1}{2}\mathrm{Tr}(N-M)\right]^2 + \det h. \qquad (8)$$

The ten parameters of T are all functions (lattice functions) of the coordinate s around the closed orbit and are written as

Six principal lattice functions:
$\alpha_u\ \beta_u\ \phi_u$ (from A)
$\alpha_v\ \beta_v\ \phi_v$ (from B)
Four coupling lattice functions:
a, b, c, d (ad - bc = 1)
$\left[\text{from } D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]$
and ψ.

For the coupling lattice functions, ψ gives the overall strength of coupling and D gives the specific mode of coupling. For example, a and d give the angles of rotation of the principal axes u and v from x and y, and b gives the fraction of the amplitudes of x and y coupled over to those of v' and u'.

3 COUPLING ELEMENTS

There are two types of coupling elements: the skew quadrupole (SQ) and the solenoid (SL). The transfer matrix across an SQ is

$$T_{SQ} = \frac{1}{2}\left(\begin{array}{c|c} 1 & 1 \\ \hline -1 & 1 \end{array}\right)\left(\begin{array}{c|c} f & 0 \\ \hline 0 & g \end{array}\right)\left(\begin{array}{c|c} 1 & -1 \\ \hline 1 & 1 \end{array}\right)$$

$$= \frac{1}{2}\left(\begin{array}{c|c} g+f & g-f \\ \hline g-f & g+f \end{array}\right), \qquad (9)$$

where

$$f = \begin{pmatrix} \cos\sqrt{q}s & \frac{1}{\sqrt{q}}\sin\sqrt{q}s \\ -\sqrt{q}\sin\sqrt{q}s & \cos\sqrt{q}s \end{pmatrix}$$

$$g = \begin{pmatrix} \cosh\sqrt{q}s & \frac{1}{\sqrt{q}}\sinh\sqrt{q}s \\ \sqrt{q}\sinh\sqrt{q}s & \cosh\sqrt{q}s \end{pmatrix}$$

and $q \equiv \frac{1}{B\rho}\frac{\partial B_x}{\partial x}$.

The transfer matrix across an SL is

$$T_{SL} = \left(\begin{array}{c|c} e & 0 \\ \hline 0 & e \end{array}\right)\left(\begin{array}{c|c} \cos ks & \sin ks \\ \hline -\sin ks & \cos ks \end{array}\right), \qquad (10)$$

where

$$e = \begin{pmatrix} \cos ks & \frac{1}{k}\sin ks \\ -k\sin ks & \cos ks \end{pmatrix} \text{ with } k \equiv \frac{1}{2}\frac{B_z}{B\rho}.$$

For "thin" elements these become

$$T = \left(\begin{array}{c|c} I & K \\ \hline -K^+ & I \end{array}\right) \text{ with } \begin{cases} K_{SQ} = \begin{pmatrix} 0 & 0 \\ qs & 0 \end{pmatrix} \\ K_{SL} = \begin{pmatrix} ks & 0 \\ 0 & ks \end{pmatrix} \end{cases}. \qquad (11)$$

If in a lattice there are a number of thin coupling elements, the one-turn transfer matrix is

$$T = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & N \end{array}\right)_{on}\left(\begin{array}{c|c} I & K \\ \hline -K^+ & I \end{array}\right)_n \cdots \left(\begin{array}{c|c} I & K \\ \hline -K^+ & I \end{array}\right)_1\left(\begin{array}{c|c} M & 0 \\ \hline 0 & N \end{array}\right)_{1o} \qquad (12)$$

$$= \left(\begin{array}{c|c} M & 0 \\ \hline 0 & N \end{array}\right)_o\left(\begin{array}{c|c} I & F \\ \hline -F^+ & I \end{array}\right),$$

where

$$F \cong \sum_i M_{oi}K_iN_{io} \text{ and } K_i = K_{SQ} \text{ or } K_{SL},$$

and $\left(\begin{array}{c|c} M & 0 \\ \hline 0 & N \end{array}\right)_o$ is the uncoupled one-turn matrix from location o around back to o. In F we have taken only linear terms in K_i of the thin (weak) coupling elements.

4 GLOBAL DECOUPLING

There are two primary global features that are affected by coupling—the betatron tunes and the emittances. Other quantities affected are derivatives of these two, such as the chromaticity.

4.1 Minimum Separation between Eigentunes

The minimum separation between eigentunes is governed by the relation given as Eq. (8) and was studied in detail in Ref. 3. It is important to note that the left-hand side of Eq. (8) is independent of s. To make it vanish, i.e., to make $\nu_u = \nu_v$ in fractional parts, one has to tune both terms on the right-hand side to zero at some location s. The equation then ensures that these terms are zero everywhere around the ring. The first term, Tr(N-M), is tuned mainly by the normal lattice quadrupoles and the second term, det h, is tuned by the skew quadrupoles. In general, two SQs are necessary to compensate all error couplings in the ring (from roll errors of quadrupoles and vertical orbit distortions in sextupoles) to give det h = 0.

For normal operation of the ring, however, $\nu_u \neq \nu_v$ in fractional parts, and the lattice can be decoupled (det h = 0) only at specific locations. In a synchrotron radiation storage ring it is desirable to be decoupled at insertion devices. Local decoupling is, thus, useful.

4.2 Emittance Coupling

Two bilinear invariants were given in Ref. 2. The relevant one here is

$$W_u = U^+J_uU = (X^+J_uX)\cos^2\psi + (Y^+D^+J_uDY)\sin^2\psi$$
$$- (X^+J_uDY + Y^+D^+J_uX)\sin\psi\cos\psi, \qquad (13)$$

where

$$J_u = \begin{pmatrix} \alpha_u & \beta_u \\ -\gamma_u & -\alpha_u \end{pmatrix}$$

and

$$X^+ = \tilde{X}\tilde{S} = \text{symplectic conjugate}.$$

The concept and formulation of the 2-dimensional emittances ε_x and ε_y for a beam of particles moving in the 4-dimensional phase space are not well defined, but the following observations are useful.

1. Setting first Y = 0, then X = 0, we get the projections on the 2-dimensional coordinate planes which, when averaged around the ring and averaged over the particles, may be defined as the emittances. This gives

$$\begin{cases} \varepsilon_x \equiv \overline{(X^+ J_u X)}\,\overline{\cos^2\psi} \\ \varepsilon_y \equiv \overline{(Y^+ D^+ J_u D Y)}\,\overline{\sin^2\psi} \end{cases} . \quad (14)$$

2. When the ring is not coupled $\psi = 0$ and we have $\varepsilon_y = 0$ and $\varepsilon_x = \overline{(X^+ J_u X)} \equiv \varepsilon_o$ = total emittance.
3. The symplectic D-transformation $D^+ J_u D$ is area preserving.

These considerations lead to the identification

$$\begin{cases} \varepsilon_x = \varepsilon_o \overline{\cos^2\psi} = \varepsilon_o (1 - \overline{\sin^2\psi}) \\ \varepsilon_y = \varepsilon_o \overline{\sin^2\psi} \end{cases} . \quad (15)$$

To measure $\overline{\sin^2\psi}$, we get from Eq. (5)

$$\frac{\overline{\det h}}{(\cos 2\pi\nu_v - \cos 2\pi\nu_u)^2} = \overline{\sin^2 2\psi} = 4\overline{\sin^2\psi}\left(1 - \overline{\sin^2\psi}\right), \quad (16)$$

where $\overline{\det h}$ can be calculated from the strengths of the two SQs after they are tuned to give zero eigen-tune separation, namely $\nu_u = \nu_v$.

5 LOCAL DECOUPLING

For local decoupling we leave the one-turn transfer matrix and concentrate on the transfer matrix over the local section of lattice, say, from -s to s. If the decoupling is to be at o, one generally needs four SQs on either side of o tuned such that F = 0 for the transfer matrix from -s to s and that, for the transfer matrix from o to s, F has an appropriate value to cancel the existing value at o. In Ref. 3 certain symmetric arrangements of the SQs were suggested to simplify the calculation and the design.

In a synchrotron radiation storage ring one generally wants the lattice to be decoupled at the straight sections for insertion devices, but the number of installed SQs is generally much less than eight per straight section. Thus, we have a least squares problem of minimizing coupling in all straight sections with a small number of available SQs. For this and for many procedures discussed earlier we need the Beam Coupling Monitor (BCM) to measure the coupling. The most direct way to measure the coupling is to impart an orbit deflection in one plane using correction dipoles and measure the orbit distortion produced in the other plane using BPMs. If the orbit deflection imparted is $\begin{pmatrix} \delta X \\ \delta Y \end{pmatrix}$ and the orbit distortion immediately downstream is written as $\begin{pmatrix} X \\ Y \end{pmatrix}$, from orbit closure we get after one revolution

$$\left(\begin{array}{c|c} M & m \\ \hline n & N \end{array}\right)\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \delta X \\ \delta Y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}. \quad (17)$$

We make two separate measurements:

1. δX only $(\delta Y = 0)$, Eq. (17) gives

$$\begin{cases} nX = (1 - N)Y \\ mY = (1 - M)X - \delta X \end{cases} \quad (18)$$

2. δY only $(\delta X = 0)$, Eq. (17) gives

$$\begin{cases} mY = (1 - M)X \\ nX = (1 - N)Y - \delta Y \end{cases} \quad (19)$$

The first ones of Eqs. (18) and (19) are independent of the specific values of the deflections and are most useful. In terms of F they are

$$\begin{cases} F^+ X = (1 - N^{-1})Y & (\delta X \text{ only}) \\ FY = -(1 - M^{-1})X & (\delta Y \text{ only}) \end{cases} . \quad (20)$$

The four scalar equations obtained by applying one each of δX and δY deflections are not independent. To get independent equations we can apply, say, two independent horizontal deflections $\delta X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\delta X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The four scalar equations so obtained from the first of Eq. (20) are independent and will determine all four elements of F from which we can calculate ψ. The whole procedure could be automated and the BCM, although not as fast as the BPM, should be able to give the readings in a few seconds.

The remainder of the procedure is identical to that for closed orbit correction. The BCM measurement is performed for each individual SQ to get the response matrix elements and some least-squares algorithm, e.g., the convenient Singular Value Decomposition procedure, is then used to minimize, say, the values of ψ at all the BCM locations.

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7 REFERENCES

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