INTRINSIC THIRD ORDER ABERRATIONS IN ELECTROSTATIC AND MAGNETIC QUADRUPOLES

R. Baartman, TRIUMF, 4004 Wesbrook Mall, Vancouver, B.C., Canada, V6T 2A3

Abstract

Intrinsic aberrations are those which occur due to the finite length of the desired field configuration. They are often loosely ascribed to the fringing field. This is misleading as it implies that the effects can be minimized by shaping the fields. In fact, there is an irreducible component related to the broken symmetry. It is present even in the hard-edge limit, and moreover, the other (soft-edge) effects can be simply ascribed to the intrinsic aberration spread over a finite length.

We rederive the aberration formulas for quadrupoles using a Hamiltonian formalism. This allows for an easy comparison of electrostatic and magnetic quadrupoles. For different combinations of large and small emittances in the two transverse planes, it is found that in some situations electrostatic quadrupoles have lower aberrations, while in others, magnetic quadrupoles are better. As well, we discuss the ways in which existing transport codes handle quadrupole fringe fields. Pitfalls are pointed out and improvements proposed.

1 INTRODUCTION

A common prescription for quadrupole design is that the beam occupy no more than a certain fraction, usually 1/2 to 3/4, of the aperture. Another common prescription is that the longer the quadrupole is compared with its bore diameter, the better. As well, it is common practice to carefully round the ends of the poles. We show that none of these common practices can be validated by the quadrupole dynamics up to 3$^{\rm rd}$ order in force.

The misconceptions are perpetuated by existing transport codes like `GIOS`[1] and `COSY`[2], because they allow one to make calculations with no fringe fields. Then when fringe fields are included, aberrations increase. In fact, the no-fringe-field cases are non-physical and such calculations should not be permitted in transport codes. The aberrations in question are not caused by the fringe fields, but by the broken symmetry inherent in a quadrupole of finite length.

We start with the quadrupole Hamiltonian and find canonical transformations for both the electrostatic and magnetic cases which eliminate the derivatives of the quadrupole strength up to 4$^{\rm th}$ order. In this way, we easily reproduce the known aberration formulas, but with additional physical insight.

2 THEORY

2.1 Electrostatic

When using the longitudinal position as independent variable, the Hamiltonian H is just the longitudinal momentum:

$$H = -\sqrt{p_0^2 - 2mq\Phi - p_x^2 - p_y^2}. \tag{1}$$

The electrostatic potential is $\Phi(x,y,z)$ and p_0 is the reference momentum. We will benefit from cleaner and more transparent notation if momenta are measured in units of p_0. This has the additional benefit that to first order, $p_x = x'$. Additionally, we let the potential Φ in units of the reference kinetic energy $p_0^2/(2m)$. Then

$$H = -\sqrt{1 - \Phi - p_x^2 - p_y^2}. \tag{2}$$

We expand the square root to $4{\rm th}$ order in coordinates and ignore the constant:

$$H \approx \frac{1}{2}(\Phi + p_x^2 + p_y^2) + \frac{1}{8}(\Phi + p_x^2 + p_y^2)^2. \tag{3}$$

To the same order, Laplace's equation gives for the expansion of the quadrupole potential:

$$\Phi = V(z)(x^2 - y^2) - \frac{V''(z)}{12}(x^4 - y^4). \tag{4}$$

The final Hamiltonian, correct to 4$^{\rm th}$ order is

$$\begin{aligned} H &= \frac{1}{2}\left[V\,(x^2 - y^2) - \frac{V''}{12}(x^4 - y^4) + p_x^2 + p_y^2\right] \\ &\quad + \frac{1}{8}\left[V\,(x^2 - y^2) + p_x^2 + p_y^2\right]^2. \end{aligned} \tag{5}$$

The trouble with applying this to simple cases like thin lenses and hard-edge limits is the presence of $V''(z)$, which becomes singular in those limits. In most cases, one sacrifices physical insight and simply traces particles with this Hamiltonian, using a more-or-less realistic function $V(z)$. For example, the approach taken in `GIOS`[1] is to leave it up to the user to specify 'fringe field integrals' such as $\int V^2 dz$ through the fringe fields. However, this leaves much room for error; different integrals may not be realistic or consistent with each other. Moreover, if one needs to solve Laplace's equation to find fringe field integrals, one might as well use the solution directly in a ray-tracing code. If one does go through this exercise, one discovers that the higher order aberrations are relatively insensitive to the 'hardness' of the quadrupole edges. This leads one to suspect that the aberrations are dominated by an intrinsic effect which has nothing to do with the detailed shape of the fringing field. Such is indeed the case.

It turns out to be possible to find a canonical transformation which eliminates the derivatives of $V(z)$. In our case, we wish to retain terms to 4$^{\mathrm{th}}$ order in the Hamiltonian (3$^{\mathrm{rd}}$ order on force), and the transformation $(x, p_x, y, p_y) \rightarrow (X, P_X, Y, P_Y)$ has generating function

$$G(x, P_X, y, P_Y) = xP_X + yP_Y + \frac{V'}{24}(x^4 - y^4) + -\frac{V}{6}(x^3 P_X - y^3 P_Y). \tag{6}$$

To the same order, this yields the transformation

$$\begin{aligned} x &= X + \frac{V}{6}X^3 \\ p_x &= P_X - \frac{V}{2}X^2 P_X + \frac{V'}{6}X^3. \end{aligned} \tag{7}$$

The y-transformation is obtained by replacing x, p_x, X, P_X with y, p_y, Y, P_Y and V with $-V$. Note that outside the quadrupole, the transformed coordinates are the same as the original ones.

This yields the transformed Hamiltonian H^*:

$$\begin{aligned} H^* &= \frac{V}{2}(X^2 - Y^2) + \frac{1}{2}(P_X^2 + P_Y^2) + \\ &+ \frac{1}{8}(P_X^2 + P_Y^2)^2 - \frac{V}{4}(X^2 + Y^2)(P_X^2 - P_Y^2) \\ &+ \frac{7V^2}{24}(X^4 + Y^4) - \frac{V^2}{4}X^2 Y^2. \end{aligned} \tag{8}$$

We can identify the terms: the first two are the usual linear ones; the third term is not related to the electric field (it is small and due to the fact that $x' \neq p_x$ or, equivalently, $\tan\theta \neq \sin\theta$); the 4$^{\mathrm{th}}$ term is also small and arises because a particle going through the quadrupole at an angle is inside the quad for slightly longer than one which remains on axis. See ref. [3] for more complete physical derivation of the individual terms.

The dominating higher order terms are the last two terms in eqn. 8. Since there are no derivatives of V, we can directly write down the aberrations in the thin-lens limit:

$$\Delta p_x = \frac{-1}{f^2 L}\left(\frac{7}{6}x^3 - \frac{1}{2}xy^2\right), \tag{9}$$

with a similar expression for Δp_y. L and f are the quadrupole's effective length and focal length. The fractional focal error is found by dividing by the linear part $\Delta_0 p_x = -x/f$:

$$\frac{\Delta f_x}{f} = \frac{1}{fL}\left(\frac{7}{6}x^2 - \frac{1}{2}y^2\right) \tag{10}$$

for x, and similarly for y.

2.2 *Magnetic*

In magnetic fields, the canonical momentum $\vec{p}$ contains the vector potential $\vec{A}$ so that the time-based Hamiltonian is

$$H_\tau = \frac{1}{2m}\left|\vec{p} - q\vec{A}\right|^2 \tag{11}$$

As before, we use the invariant $p_0 \equiv \sqrt{2mH_\tau}$ to normalize the momenta, convert to z as independent variable, and expand the square root, keeping terms up to 4$^{\mathrm{th}}$ order:

$$\begin{aligned} H &\approx -A_z + \frac{1}{2}\left[(p_x - A_x)^2 + (p_y - A_y)^2\right] + \\ &+ \frac{1}{8}\left[(p_x - A_x)^2 + (p_y - A_y)^2\right]^2. \end{aligned} \tag{12}$$

To this order, the vector potential for quadrupole strength $k(z)$ is

$$\begin{aligned} A_x = -\frac{k'}{4}xy^2, \quad A_y = \frac{k'}{4}x^2 y \\ A_z = -\frac{k}{2}(x^2 - y^2) + \frac{k''}{48}(x^4 - y^4), \end{aligned} \tag{13}$$

and the Hamiltonian can be written:

$$\begin{aligned} H &= \frac{1}{2}\left[k\,(x^2 - y^2) - \frac{k''}{24}(x^4 - y^4) + p_x^2 + p_y^2\right] + \\ &+ \frac{k' xy}{4}(yp_x - xp_y) + \frac{1}{8}(p_x^2 + p_y^2)^2. \end{aligned} \tag{14}$$

The generating function which will eliminate derivatives of k is

$$G(x, P_X, y, P_Y) = xP_X + yP_Y + \frac{k'}{48}(x^4 - y^4) + -\frac{k}{12}\left[(x^3 + 3xy^2)P_X - (3x^2 y + y^3)P_Y\right], \tag{15}$$

which, to the same order yields transformation

$$x = X + \frac{k}{12}(X^3 + 3XY^2) \tag{16}$$

$$p_x = P_X - \frac{k}{4}\left[(X^2 + Y^2)P_X - 2XYP_Y\right] + \frac{k'}{12}X^3,$$

and similarly for (y, p_y). The transformed Hamiltonian is

$$\begin{aligned} H^* &= \frac{k}{2}(X^2 - Y^2) + \frac{1}{2}(P_X^2 + P_Y^2) + \\ &+ \frac{1}{8}(P_X^2 + P_Y^2)^2 - \frac{k}{4}(X^2 + Y^2)(P_X^2 - P_Y^2) \\ &+ \frac{k^2}{12}(X^4 + Y^4) + \frac{k^2}{2}X^2 Y^2. \end{aligned} \tag{17}$$

Notice the similarity to eqn. 8: in fact all terms are identical except the last two, which only differ in their coefficients. Applying the same procedure as in the electrostatic case, we write down the fractional change in focusing strength:

$$\frac{\Delta f_x}{f} = \frac{1}{fL}\left(\frac{1}{3}x^2 + y^2\right) \tag{18}$$

3 DISCUSSION

Formulas 10 and 18 are handy for quickly evaluating the importance of 3rd order aberration. They also show that for fixed focal length, the **only** way of reducing the aberration is by lengthening the quadrupole; the fraction of aperture used is not important; for a given effective length, the absolute size of the aperture is not important; the shape of the ends of the electrode is not important.

Comparing the two formulas, we see that for roundish beams ($x \approx y$), electrostatic and magnetic quads yield similar aberrations: they are in the ratio of $\frac{7}{6} : \frac{4}{3}$. For cases where one transverse dimension is large compared with the other, and it is important to maintain the quality in the larger dimension, magnetic quads are better by a factor of $\frac{7}{2}$. However, for the more common case where it is more important to maintain the quality of the higher quality dimension, electrostatic quads win by a factor of 2.

Results from using the above Hamiltonians are in agreement with those from using the commonly used codes `GIOS` and `COSY`, provided fringe field cards are used. In both of those codes it is possible to perform a 3rd order calculation with quads which have no fringe fields. This gives incorrect and actually completely unphysical results. In essence, omitting the fringe field cards in those codes describes a situation where the particle traverses non-Maxwellian fields. For example, `GIOS`, since it does not use the scalar value of the potential field, does not obey conservation of energy when fringe field cards are omitted.

The hard-edge case is correctly described in `GIOS` by including fringe field cards and setting the quadrupole aperture to zero, or, equivalently, setting all the fringe field integrals to zero. This is a useful approximation since the results are usefully close to reality and yet one needs not worry about specifying realistic fringe field integrals. This does not work in `COSY`, since a zero aperture forces an infinitesimal integration step-size. A better solution would be to build in the hard-edge kicks and use these as default when no fringe field is specified.

The required hard-edge kicks at the entrance to the quadrupole are derived directly from equations 7 and 16. The reason is that we know that the transformed coordinates (X, P_X, Y, P_Y) do not experience any singular forces in the hard-edge limit. Therefore, the kicks for those coordinates are all zero. So the kicks for the untransformed (x, p_x, y, p_y) for the electrostatic case are,

$$\begin{aligned} \Delta x &= \frac{V}{6}x^3 \\ \Delta p_x &= \frac{-V}{2}x^2 p_x \\ \Delta y &= \frac{-V}{6}y^3 \\ \Delta p_y &= \frac{V}{2}y^2 p_y, \end{aligned} \tag{19}$$

and for the magnetic case are,

$$\begin{aligned} \Delta x &= \frac{k}{12}(x^3 + 3xy^2) \\ \Delta p_x &= \frac{-k}{4}\left[(x^2+y^2)p_x - 2xyp_y\right] \\ \Delta y &= \frac{-k}{12}(3x^2y + y^3) \\ \Delta p_y &= \frac{k}{4}\left[(x^2+y^2)p_y - 2xyp_x\right]. \end{aligned} \tag{20}$$

The kicks at the exit are, of course, opposite in sign. These agree with the `GIOS` case of zero fringe field integrals. See ref. [1].

4 REFERENCES

[1] H. Matsuda and H. Wollnik *Third Order Transfer Matrices for the Fringing Field of Magnetic and Electrostatic Quadrupole Lenses* NIM **103**, p. 117 (1972).

[2] M. Berz *Computational Aspects of Design and Simulation: COSY INFINITY* NIM **A298**, p. 473 (1990).

[3] R. Baartman *Aberrations in Electrostatic Quadrupoles* TRI-DN-95-21 (TRIUMF internal note, 1995).