# LANDAU DAMPING OF THE WEAK HEAD–TAIL INSTABILITY FOR BEAMS WITH QUADRATIC AMPLITUDE–DEPENDENT BETATRON TUNES AND BINOMIAL AMPLITUDE DISTRIBUTIONS

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*Abstract*

The simple theory of the weak head–tail instability does not exhibit a threshold current. Experimentally, the instability does not always appear, due to the readily available stabilizing mechanism known as Landau damping. Here we extend the theory to include Landau damping that arises from a quadratic dependence of betatron tune on amplitude, as occurs when octupoles are present in the synchrotron lattice. The binomial density distribution, which is appropriate for proton beams, is chosen as the steady–state for both the transverse and longitudinal phase–space distributions. Using Sacherer's formalism we look for solutions by expanding the perturbation in a set of basis functions. The method involves evaluating the dispersion integral, and finding the eigen–values of the interaction matrix. The results can be represented in a "stability diagram" with the standard interpretation but with the axes representing the real and imaginary parts of the eigen–values. As a numerical example, we determine the octupole strength needed to damp the reported high–order modes observed[1] in the CERN PS for bunches with "LHC characteristics".

## 1 INTRODUCTION

With ever increasing intensity, Landau damping of instabilities by the frequency spread produced by the residual nonlinearity in the transverse restoring force cannot be assumed to be sufficient, especially at high energies where the image charge force becomes negligible. This is clearly demonstrated by the observation of the high–order modes $l = 5, 6,$ and 7 at the CERN PS [1].

The simple theory of the instability is formulated under the assumption that there is no intrinsic spread in the incoherent oscillation frequencies. With an intrinsic spread, coherent modes can exist only above a certain threshold and their frequencies are called Landau frequencies. We have extended the theory of Sacherer[2] to include Landau damping by a betatron frequency spread. We have derived an integral equation[3] which contains a dispersion integral due to the amplitude dependent betatron frequency; we call it the dispersive integral equation. The equation reduces to Sacherer's integral equation when there is no frequency spread, as expected.

In this report, we will consider a specific case in which the amplitude dependence is quadratic, as occurs when octupoles are present in the ring. The amplitude distributions of the transverse and longitudinal steady–state distributions are chosen to be the binomial distribution because its profile is sharp, which is appropriate for proton beams.

## 2 THE DISPERSION RELATION

The dispersive integral equation below is obtained from the linearized Vlasov equation where we have neglected transverse and longitudinal mode coupling and assumed that the transverse wake force does not affect the synchrotron motion,

$$\frac{R_l(r)}{f(\Lambda) + i\,g(\Lambda)} = W(r) \int_0^\infty r'\,dr' R_l(r') G_l(r, r'), \quad (1)$$

where $W(r)$ is the weight function, $R_l(r)$ the longitudinal radial density function of the $l$th head–tail mode, $G_l(r, r')$ the kernel function of the integral, and $\Lambda$ the Landau mode frequency. The beam transfer response function (BTRF) $f(\Lambda) + i\,g(\Lambda)$ is obtained by evaluating the dispersion integral,

$$f(\Lambda) + ig(\Lambda) = \frac{1}{2} \int_0^\infty \frac{-f_0'(q)q^2\ dq}{\Lambda - \omega_\beta(q) - l\omega_s}, \quad (2)$$

where $f_0(q)$ is the binomial transverse amplitude $(q)$ distribution and $\omega_s$ is the synchrotron frequency.

We followed Sacherer's formalism in transforming the integral equation into a set of linear equations by expanding $R_l(r)$ in terms of an appropriate set of basis functions with $W(r)$ as the weight,

$$\left[\frac{1}{f(\Lambda) + i\,g(\Lambda)}\right]\ a_k = \sum_{k'} M_{kk'} a_{k'}, \quad (3)$$

where $k$ and $k'$ are the index of the basis function ($f_k(r),\ k = 0, 1, 2, 3, \ldots$), and $a_{k'}$ the coefficients of the expansion. The matrix elements $M_{kk'}$ are given by a summation of the total transverse impedance $Z_1(\omega')$ times the frequency spectra $h_{lk}(\omega')$ and $h_{lk'}(\omega')$ of the phase–space density functions formed by the $k$th and $k'$th basis functions times the phase density distribution,

$$M_{kk'} = -i\frac{\pi N r_0 c}{\gamma T_0^2 \omega_\beta} \sum_p Z_1(\omega') h_{lk}(\omega') h_{lk'}(\omega'), \quad (4)$$

where $N$ is the number of particles in the bunch, $r_0$ the classical proton (electron) radius, $T_0$ the revolution period, $\omega_\beta$ the small amplitude betatron frequency, $\gamma$ and $c$ have the usual relativistic definitions. The summation is over the side bands of the revolution frequency $\omega_0$; $\omega' = p\omega_0 + \Lambda$. For a nontrivial solution to exist, the reciprocal of the BTRF must satisfy

$$\det\left[\left(\frac{1}{f(\Lambda) + i\,g(\Lambda)}\right)\mathbf{I} - \mathbf{M}\right] = 0, \quad (5)$$

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where $\mathbf{I}$ is the identify matrix and $\mathbf{M}$ is the interaction matrix defined above. This is true if and only if

$$\frac{1}{f(\Lambda)+i\,g(\Lambda)}=\lambda_n, \quad n=0,1,2,3\ldots, \qquad (6)$$

where $\lambda_n$ is the $n$th eigen–value of $\mathbf{M}$. Thus, there are an infinite set of dispersion relations for each mode $l$, one for each eigen–value.

In the absence of an intrinsic frequency spread, the dispersion relation Eqn. 6 reduces to an expression which can be rearranged to give the Sacherer frequencies $\Omega$ of the $l$th mode,

$$\Omega=\lambda_n+l\omega_s+\omega_\beta. \qquad (7)$$

The result above is obtained by evaluating the BTRF for the special case of no frequency frequency spread [3], that is, $\omega_\beta(q)=\omega_\beta$.

The dispersion relation Eqn. 6 has very limited usage. According to the standard interpretation of the dispersion relation, and using the definition of the Sacherer frequency (Eqn. 7), we can construct a complex map of the Landau frequency plane $(\Lambda)$ into the Sacherer frequency plane $(\Omega)$ from the BTRF as follows,

$$\frac{1}{f(\Lambda)+i\,g(\Lambda)}=\Omega-l\omega_s-\omega_\beta. \qquad (8)$$

This complex map allows one to obtain the stability boundary diagram in the familiar Sacherer plane by mapping contour lines of constant growth rates. The Landau damped (stable) region is free of contour lines and bounded by the line of zero growth and the horizontal axis. See Fig. 1. An unstable mode is Landau damped if its Sacherer frequency lies in this region. Ideally, one wants to obtain the stability diagram in the transverse impedance $Z_1(\omega)$ plane. However, it must be pointed out that the transverse impedance cannot be, at best, practically obtained from the Sacherer frequency.

## 3 THE BEAM TRANSFER RESPONSE FUNCTION

As an example, we will obtain the BTRF of a binomial amplitude distribution with quadratic amplitude dependent betatron frequency. The quadratic dependence can be written as

$$\omega_\beta(q)=\omega_\beta+Sq^2/\hat{q}^2, \qquad (9)$$

where $0\le q\le\hat{q}$, $\hat{q}$ is the maximum betatron amplitude, and $S$ the width of the frequency spread. We use the binomial distribution of the form,

$$f_0(q)=\frac{\alpha(1+\alpha)}{2\hat{q}^2}\left(1-\frac{q^2}{\hat{q}^2}\right)^\alpha, \qquad (10)$$

where $\alpha$ is a parameter that determines the profile of the distribution. We cannot evaluate the dispersion integral for an unspecified value of $\alpha$. We choose a specific value of 4 so we can use the result later. Substituting the binomial distribution and the quadratic dependence into the dispersion integral (Eqn. 2) and performing the integration, we obtain the BTRF,

$$\begin{aligned} f(\Lambda) &= \frac{20}{S^2}\left(\frac{\Delta\Lambda}{3}-\frac{3S}{4}+\frac{3S^2-3S\Delta\Lambda+\Delta\Lambda^2}{2S}\right.\\ &+\frac{(\Delta\Lambda-S)^3}{S^2}+\frac{\Delta\Lambda(\Delta\Lambda-S)^3\log|S-\Delta\Lambda|}{S^3}\\ &+\left.\frac{(S-\Delta\Lambda)^3\Delta\Lambda\log|\Delta\Lambda|}{S^3}\right),\\ g(\Lambda) &= \frac{20\pi}{S^2}\left(\frac{\Delta\Lambda(\Delta\Lambda-S)^3}{S^3}\right), \end{aligned} \qquad (11)$$

where $\Delta\Lambda=\Lambda-\omega_\beta-l\omega_s$.

## 4 THE HEAD–TAIL MODE SPECTRA

The mode spectra are required in the evaluation of the matrix elements of $\mathbf{M}$. In this section, we will outline the steps involved in obtaining the mode spectra $h_{lk}(\omega)$ of the longitudinal binomial amplitude distribution from the basis functions $f_k(r)$. The normalized longitudinal distribution $g_0(r)$ is

$$g_0(r)=\frac{(1+\alpha)}{\pi\hat{r}^2}\left(1-\frac{r^2}{\hat{r}^2}\right)^\alpha, \qquad (12)$$

where $\hat{r}$ is the maximum synchrotron amplitude, and the orthogonality condition that defines the basis functions is

$$\int_0^{\hat{r}} r\,dr\,g_0(r)f_k(r)f_{k'}(r)=\delta_{kk'}, \qquad (13)$$

where $\delta_{kk'}$ is the Kroneker delta function. Looking up a standard list of weighted orthonormal functions we found the correct set and, after the normalization, we can write the basis functions as

$$\begin{aligned} f_k(r) &= \sqrt{\frac{2\pi(1+\alpha+2k+l)\Gamma(1+\alpha+k+l)\,k!}{(1+\alpha)\Gamma(1+k+l)\Gamma(1+\alpha+k)}}\\ &\times \left(\frac{r}{\hat{r}}\right)^l P_k^{l,\alpha}\left(1-2\frac{r^2}{\hat{r}^2}\right), \end{aligned} \qquad (14)$$

where $\Gamma(x)$ is the gamma function and $P_k^{l,\alpha}(x)$ is the Jacobbi polynomial of order $k$ with indices $l$ and $\alpha$. From the basis functions, we formed the corresponding phase–space density functions and took their Fourier transforms to obtain the spectra,

$$\begin{aligned} h_{lk}(\omega) &= 2^\alpha\sqrt{\frac{2}{\pi}\frac{(1+\alpha)(1+\alpha+2k+l)}{\Gamma(1+k+l)\,k!}} \qquad (15)\\ &\times \sqrt{\Gamma(1+\alpha+k+l)\Gamma(1+\alpha+k)}\\ &\times J_{1+\alpha+2k+l}\left(\frac{\omega-\omega_\xi}{c}\hat{r}\right)\Big/\left(\frac{\omega-\omega_\xi}{c}\hat{r}\right)^{1+\alpha}, \end{aligned}$$

for $\alpha>-1$, and where $\omega_\xi$ is the chromatic frequency shift, and $J_n(x)$ the $n$th order Bessel function of the first kind.

## 5 GROWTH RATES AND FREQUENCY SHIFTS OF THE "LHC" BUNCH

In this section, we will calculate, assuming the longitudinal amplitude distribution is binomial, and attempt to compare with the experimental observation. The approximate

| $l$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 10 |
|---|---|---|---|---|---|---|---|---|
| $k$ | -120 | -45 | -7 | 5 | 5.6 | 3.6 | 2.1 | 0.7 |
| $\Delta f$ | -160 | -109 | -67 | -36 | -18 | -10 | -6 | -27 |

Table 1: Transverse head–tail frequency shifts and growth rates of the "LHC " bunch.

eigen–values can be found easily, if the cross–coupling terms, which are generally two orders of magnitude smaller than the diagonal terms, are neglected. This gives a simple expression for the frequencies,

$$\Omega_n = M_{nn} + l\omega_s + \omega_\beta. \quad (16)$$

The resistive wall impedance was identified as responsible. Using the beam parameters in [1] and also the impedance formula, we calculated $\Omega_0$ for modes 2 to 10. For each mode, only the frequency with index $n$=0 was calculated, since this index value gives the largest frequency shift ($\Delta f$ Hz) and growth rate ($k$ s$^{-1}$), roughly two orders of magnitude larger than that of the next index value ($n$=1). We evaluated the matrix element $M_{00}$ by broad–band approximating the summation with an integral, which is a good approximation since the resistive wall impedance is a smooth function without sharp peaks.

The results are shown in Table 1. We can see that modes 5,6, and 7 are unstable and they have the largest growth rates. The rise times of these three modes range from 200–285 ms, which are close to the observed[1] rise times 100–200 ms. However, we are not able to compare the frequency shifts with measurements as there are no data available. When we compared with the rise times obtained using the approximate mode spectra in Ref. [1], they differ by as much as a factor of two. This shows that the approximate spectra are inaccurate and the exact spectra $h_{lk}(\omega)$ should be used instead if a bunch has the binomial profile.

## 6 LANDAU DAMPING BY OCTUPOLES

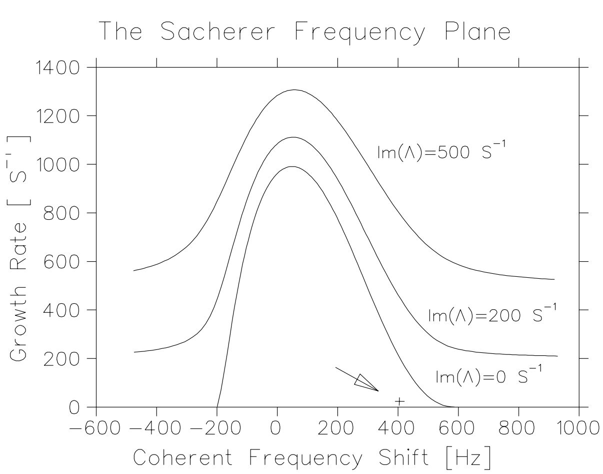


Figure 1: Contour lines and stability boundary diagram for the "LHC" bunch.

In this section, we will estimate the upper bound of the octupole strength required to Landau damp the instability as observed in [1]. Let us assume that the transverse amplitude distribution of the bunch is binomial with an $\alpha$ value of 4. This value gives the binomial distribution a profile that is similar to the Gaussian distribution but without the infinite tail. It is appropriate for bunches which do not have very sharp parabolic profiles. With octupoles installed in a ring, the time averaged betatron frequency depends quadratically on the amplitude. The BTRF for these conditions was evaluated as an example in Sec. 3 (Eqn. 11). The map can be obtained by taking the reciprocal of Eqn. 11 and equating it to the left hand side of Eqn. 6.

Given a frequency spread $S$, the contour lines are obtained by plotting the complex values of the map for different constant values of the growth rate [Im($\Lambda$)], as the frequency shift [Re($\Delta\Lambda$)] is scanned from $-\infty$ to $\infty$. An upper bound on the octupole strength can be obtained from a minimum frequency spread required to put the point whose coordinates comprises the largest growth rate and coherent frequency shift in the stable area. As reported, the largest growth rate was 10 s$^{-1}$. The frequency shifts of the unstable modes are not available. Fortunately, that of the dipole mode ($l$=0) is given [1] and it is the upper limit, which would give a conservative estimate. Guided by the well–known rule of thumb for Landau damping, we found that a frequency spread of 1000 Hz is sufficient. Fig. 1 shows the location of the point and well inside the boundary. This frequency spread requires an integrated octupole strength of 63 m$^{-3}$.

## 7 CONCLUSIONS

We have derived the dispersion relations for the weak transverse head–tail modes of the binomial amplitude distribution, which also defines a complex map. When there is no intrinsic frequency spread, it reduces to an expression for the mode frequencies in terms of the eigen–values of the interaction. We have obtained an analytic expression for the mode spectra and calculated an example BTRF. As an application, we calculated the growth rates and coherent frequency shifts of the "LHC" bunch and estimated the octupole strength to promote Landau damping.

## 8 REFERENCES

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