MICROWAVE INSTABILITY THRESHOLDS

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Abstract

Taking the first orthogonal polynomials in the conventional radial mode expansion in the eigenvalue type perturbation approach, the usual Keil-Schnell criteria for the microwave instabilities can be obtained. In this way, a close relationship between the two approaches is established. The existing results are reviewed, and some comments and modifications are made.

1 INTRODUCTION

A brief review of beam instability analyses shows that its development either belongs to a Vlasov-equation-evolved perturbation approach, or belongs to a Keil-Schnell-criterion type approach. In the first approach, see [1] and the references therein, both azimuthal and radial expansions are used to explore the particle distribution evolutions. Current direction is to include the potential well deformation, see for example [2], and to include the effect of Landau damping, see for example [3]. The development is unlikely to give rise to analytical solutions that can be easily used. On the other hand, the second approach uses crude beam profile (with an exception for the longitudinal coasting beams) to estimate the instability threshold for both bunched and coasting beams. General results can be found in [4] and the references therein. These results have been proved very useful and often provide guidance to the development and improvement of accelerators. The crude beam profile, however, has certainly imposed limitations in the application.

In this report, we show that the use of the first orthogonal polynomials in the perturbation approach can give rise to identical results obtained by the Keil-Schnell type criteria. This is owing to the fact that, in general, the first orthogonal polynomial represents the most prominent radial mode. In this way, a close relationship between the two approaches is established. Therefore, comments will be made regarding to the limitation and possible error in the applications of the simplified criteria. Some modifications will then be developed, if necessary.

2 TRANSVERSE INSTABILITY

Using the first orthogonal polynomial for the azimuthal mode $m = 0$, setting $\beta \approx 1$, the bunched beam dynamic equation becomes,

$$\omega - \omega_\beta = \frac{jeI_0}{2Rm_0\gamma\omega_\beta} \sum_{n=-\infty}^{\infty} Z_T(n)\Lambda_{0,1}^2(n') \qquad (1)$$

where ω_β is the betatron frequency, R is the machine radius. The average beam current is $I_0 = Ne\omega_0/2\pi$, where N is the number of particles, and ω_0 is the revolution frequency. Also $Z_T(n)$ is the transverse impedance, and $\Lambda_{0,1}(n)$ is the spectrum of the first orthogonal polynomial for $m = 0$, where n represents the effective spectrum line, r represents the radial position. The notation n' denotes the chromatic effect. The equivalence $\sum_{n=-\infty}^{\infty} Z_T(n)\Lambda_{0,1}^2(n') = \sum_{n=-\infty}^{\infty} Z_T(n'')\Lambda_{0,1}^2(n)$ is used in this article, where n'' denotes the frequency shift equals n' but in the opposite direction.

Consider the normalized Gaussian distribution in phase space,

$$\psi_0(r) = \frac{2}{\pi r_\ell^2} e^{-2r^2/r_\ell^2} \qquad (2)$$

where r_ℓ is the half bunch length in radius. The transverse weight function is defined as $W_T(r) = \psi_0(r)$.

In the following, the instability threshold will be obtained by the rule of thumb, which is,

$$|\Delta\Omega| < \Delta\omega \qquad (3)$$

where $\Delta\Omega$ is the coherent frequency shift, and $\Delta\omega$ is the rms or the half width of half maximum frequency spread.

2.1 Bunched Beam

An estimate of the bunched beam instability threshold can be obtained using $\Lambda_{0,1}^2(n') \approx 1/2\pi$ in (1),

$$\left| \sum_{n=-\infty}^{\infty} Z_T(n) \right| < \frac{4\pi R m_0 \gamma \omega_\beta}{eI_0} \Delta\omega \qquad (4)$$

The criterion given in the equation (5.62) of [4] can be written as,

$$\left| \sum_{n=-\infty}^{\infty} Z_T(n) \right| < \frac{2\omega_\beta \gamma T_0^2}{N r_0 c} \Delta\omega \qquad (5)$$

Using $r_0 = e^2/m_0c^2$, $T_0 = 2\pi/\omega_0$, $\omega_0 = \beta c/R$, the equation (5) becomes identical to (4).

For a long bunch with a narrow spectrum, the error of using (4) can be large, mainly owing to the use of $\Lambda_{0,1}^2(n') \approx 1/2\pi$, the peak of the power spectrum. Also the chromatic effect can introduce uncertainties.

For an improved estimate, therefore, we need to use,

$$\Lambda_{0,1}(n) = \frac{1}{\sqrt{2\pi}} e^{-n^2 r_\ell^2/8} \qquad (6)$$

Substituting (6) into (1), and considering the chromatic effect, we get,

$$\left| \sum_{n=-\infty}^{\infty} Z_T(n'') e^{-n^2 r_\ell^2/4} \right| < \frac{4\pi R m_0 \gamma \omega_\beta}{eI_0} \Delta\omega \qquad (7)$$

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- A criterion is given in [5], which can be written as,

$$\left|\bar{Z}_T(n'')\right| < \frac{2\omega_\beta\gamma E_0 z_L}{ec^2 I_0}\Delta\omega \tag{8}$$

where z_L is the full bunch length, and $\bar{Z}_T(n'')$ is the averaged impedance over the width of the bunch spectrum. The summation on the right side of (1) can be approximately taken as,

$$\sum_{n=-\infty}^{\infty} Z_T(n)\Lambda_{0,1}^2(n') \approx \bar{Z}_T(n'')\frac{2R}{z_L} \tag{9}$$

Substituting (9), using $m_0 = E_0/c^2$, the equation (1) becomes

$$\left|\bar{Z}_T(n'')\right| < \frac{\omega_\beta\gamma E_0 z_L}{ec^2 I_0}\Delta\omega \tag{10}$$

Which differs from (7) by a factor of 0.5.

- A better formalism is presented in the equation (18) in [6]. With $m = 0$, it can be written as,

$$\left|\frac{\sum_{n=-\infty}^{\infty} Z_T(n)h_0(n')}{\sum_{n=-\infty}^{\infty} h_0(n')}\right| < \frac{2\omega_\beta\gamma m_0 z_L}{eI_0}\Delta\omega \tag{11}$$

The left side is called the *effective impedance*, where $h_0(n')$ is the power spectrum of the bunch. If only the first orthogonal polynomial is used, we have $h_0(n') = \Lambda_{0,1}^2(n')$. The redundancy in the equation (11) involving the effective impedance is shown as the follows. Using $r_\ell = z_L/2R$, we can write,

$$\sum_{n=-\infty}^{\infty} h_0(n) = 2\int_0^\infty \frac{1}{2\pi}e^{-n^2 r_\ell^2/4}dn \approx \frac{2R}{\sqrt{\pi}z_L} \tag{12}$$

Applying this equation into (11), the bunch length z_L is cancelled. Since the information of the bunch length has been represented by the bunch spectrum $h_0(n')$ in the numerator of the effective impedance, this triple representation of the bunch length can be seen as redundancy. In comparison, the use of the total effective impedance shown in the left side of (7) seems to be more straightforward. Substituting (12) into (11), we get,

$$\left|\sum_{n=-\infty}^{\infty} Z_T(n'')e^{-n^2 r_\ell^2/4}\right| < \frac{8\sqrt{\pi}Rm_0\gamma\omega_\beta}{eI_0}\Delta\omega \tag{13}$$

This differs from (7) by a factor of 1.13.

2.2 Coasting Beam

For a coasting beam, the power spectrum of the perturbation is a delta function at a frequency n_1 with an amplitude $1/2\pi$. The equation (1), therefore, is modified as,

$$\omega - \omega_\beta = \frac{jeI_0}{2Rm_0\gamma\omega_\beta}\sum_{n=-\infty}^{\infty} Z_T(n)\frac{\delta(n-n_1)}{2\pi} \tag{14}$$

then the instability threshold can be estimated as,

$$|Z_T(n_1)| < \frac{4\pi Rm_0\gamma\omega_\beta}{eI_0}\Delta\omega \tag{15}$$

The criterion given in the equation (5.91) of [4] is,

$$|Z_T(n)| < \frac{2\omega_\beta\gamma T_0^2}{Nr_0 c}\Delta\omega \tag{16}$$

which is identical to the equation (15).

- The equation (4) in [5] can be written,

$$|Z_T(n)| < \frac{8FE_0\gamma\omega_\beta}{eI_0 R\omega_0^2}\Delta\omega \tag{17}$$

which can be written as,

$$|Z_T(n)| < \frac{8FRm_0\gamma\omega_\beta}{eI_0}\Delta\omega \tag{18}$$

Taking $F = 1$, this equation differs from (15), which is less tight, by a factor of 0.64.

3 LONGITUDINAL INSTABILITY

Using the first orthogonal polynomial, for the $m = 1$ mode, the longitudinal beam dynamic equation in [1] becomes,

$$\omega - \omega_S = \frac{j2\pi\omega_S I_0}{V\cos\phi_S}\sum_{n=-\infty}^{\infty}\frac{Z_L(n)}{n}\Lambda_{1,1}^2(n) \tag{19}$$

where ω_S is the synchrotron frequency, and ϕ_S is the synchronous phase, V is the RF gap voltage per ring, and $Z_L(n)/n$ is the longitudinal impedance. For a Gaussian distribution with the half bunch length r_ℓ, the longitudinal weight function is,

$$W_L(r) = -\frac{\partial\psi_0(r)}{\partial r}\frac{1}{r} = \frac{8}{\pi r_\ell^4}e^{-2r^2/r_\ell^2} \tag{20}$$

3.1 Bunched Beam

Using the equations (19) and the approximation,

$$\Lambda_{1,1}(n) \approx \frac{n}{2\sqrt{\pi}} \tag{21}$$

the bunched beam instability threshold is written as,

$$\left|\sum_{n=-\infty}^{\infty} nZ_L(n)\right| < \frac{2V\left|\cos\phi_S\right|}{\omega_S I_0}\Delta\omega \tag{22}$$

where $\Delta\omega$ is the synchrotron frequency spread. The corresponding equation (5.69) in [4] is,

$$\left|\sum_{n=-\infty}^{\infty} nZ_L(n)\right| < \frac{2\omega_S\gamma(2\pi R)^2}{Nr_0|\eta|c^2\omega_0}\Delta\omega \tag{23}$$

Using $\omega_S^2 = -\omega_0^2 e\eta V\cos\phi_S/2\pi E$, the equation (23) is shown to be the same as (22).

This criterion is indeed very crude, owing to that in arriving (21), the approximation of the Bessel function $J_1(nr) \approx nr/2$ is used, which is only valid in a small range $nr < 1$.

An improvement to this criterion, therefore, can be made by using,

$$\Lambda_{1,1}(n) = \frac{n}{2\sqrt{\pi}} e^{-n^2 r_\ell^2/8} \tag{24}$$

Substituting (24) into (19), we get,

$$\underline{\left| \sum_{n=-\infty}^{\infty} n Z_L(n) e^{-n^2 r_\ell^2/4} \right| < \frac{2V\,|\cos\phi_S|}{\omega_S I_0} \Delta\omega} \tag{25}$$

- The result in the equation (5) in [7] can be written, for $m = 1$ and the harmonic number $h = 1$, as,

$$\left| \frac{\sum_{n=-\infty}^{\infty} (Z_L(n)/n)\, h_1(n)}{\sum_{n=-\infty}^{\infty} h_1(n)} \right| < \frac{6B^3 V\,|\cos\phi_S|}{\omega_S I_0} \Delta\omega \tag{26}$$

  where $h_1(n) \approx \Lambda_{1,1}^2(n)$ is the power spectrum of the bunch, and $B = r_\ell/\pi$ is the bunching factor. Note that we have,

$$\sum_{n=-\infty}^{\infty} h_1(n) = 2\int_0^{\infty} \frac{n^2}{4\pi} e^{-n^2 r_\ell^2/4} dn \approx \frac{1}{\pi^{7/2} B^3} \tag{27}$$

  Substituting (27) into (26), using (24), we get,

$$\left| \sum_{n=-\infty}^{\infty} n Z_L(n) e^{-n^2 r_\ell^2/4} \right| < \frac{1.37V\,|\cos\phi_S|}{\omega_S I_0} \Delta\omega \tag{28}$$

  which differs from (25) by a factor of 0.69. Again we consider that the use of the equation (25) is more straightforward than (26) with the effective impedance.

3.2 Coasting Beam

The Landau damping in the longitudinal coasting beam is the most explored one. Together with the dispersion relation, the stability diagram can be plotted on the real and imaginary impedance plane. Compared with the others, this is the only case that no external focusing presented, therefore, one may expect that this case should be completely different from the others.

The successful application of the coasting beam instability criterion to the bunched beams, i.e. the Boussard criterion, has opened the door to think that at least for the long bunches and/or strong instability, the effect of the synchrotron focusing is not irreplaceable. It is found that using an equivalence

$$\omega_S \approx \Delta\omega \tag{29}$$

and the local current, the bunched beam criteria is closely related with the coasting beam criteria. To establish the relation, using,

$$\frac{1}{|\eta|}\frac{\Delta\omega}{\omega_0} = \left(\frac{\Delta p}{p}\right)_{rms} = \frac{\omega_S r_\ell}{2\,|\eta|\,\omega_0} \tag{30}$$

we find that the equation (29) is equivalent to $r_\ell = 2$. For this case, the beam power spectrum is still a delta function, but the amplitude is no longer constant. Since the amplitude of a delta function equals the area of the function, i.e. $\sum_{n=-\infty}^{\infty} \Lambda_{1,1}^2(n)$, removing the impedance $Z_L(n)/n$ out of the summation on the right side of (19), we get,

$$\left| \frac{Z_L(n)}{n} \right| < \frac{E r_\ell^4}{2\sqrt{2} e |\eta| \omega_0^2 I_0} (\Delta\omega)^2 \tag{31}$$

Substituting $r_\ell = 2$, the equation (31) becomes,

$$\underline{\left| \frac{Z_L(n)}{n} \right| < \frac{5.66E}{e|\eta|\omega_0^2 I_0} (\Delta\omega)^2} \tag{32}$$

The Keil-Schnell criterion shown in the equation (5.131) in [4] can read,

$$\left| \frac{Z_L(n)}{n} \right| < \frac{0.68\gamma T_0^3}{2\pi N r_0 |\eta|} (\Delta\omega)^2 \tag{33}$$

where the tri-elliptical spectrum is used. This equation can be written the same as (32), except that the factor 5.66 becomes 4.27.

- The equation (1) in [5] can be written,

$$\left| \frac{Z_L(n)}{n} \right| < \frac{F\gamma E_0 |\eta|}{e I_0} \left(\frac{\Delta p}{p}\right)^2 \tag{34}$$

  where the form factor F is a unity. Since $\Delta p/p$ is the full momentum spread at half height, using $\Delta p/p = 2(\Delta p/p)_{rms}$, the equation (34) is the same as (32), except that the factor 5.66 becomes 4.

4 REFERENCES

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