NEW METHOD FOR POINT-CHARGE WAKEFIELD CALCULATION*

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Abstract

Extending our approach recently described in [1] we present a new method to accurately calculate point-charge geometric wakefields from wake potentials due to a much longer bunch, typically obtained with a time-domain EM field solver. By allowing a relatively long bunch in the EM solver, this method can significantly reduce the need for computer resources as well as drastically shorten the computing time. On top of that, the method provides valuable physics insights. Since the method, applied to longitudinal wakes, was described in [1], the goal of this paper is to extend it to transverse wakes.

METHOD DESCRIPTION

Knowledge of geometric wakefields is critically important for studies of accelerator beam dynamics. While analytical solutions are known for a number of simple geometries, detailed wakefield calculations for realistic vacuum chamber components are typically done utilizing time domain EM solvers. They compute the fields due to finite length bunches, forcing one to use extremely fine mesh (small fraction of the bunch length) to compute wakes at small distances. This is where the longitudinal wakes are usually dominated by singularities, so that a wake potential due a bunch of rms length σ scales as $W_{\parallel}^{\sigma}(z)\propto\sigma^{-q}$, q>0 [2,3]. Utilizing fine meshes has severe implications for computer memory requirements as well as calculation speed. Furthermore, EM solvers cannot calculate point-charge wakes, $W_{\parallel}^{\delta}(z)$ and $W_{\perp}^{\delta}(z)$, - functions that often provide significant physics insights and simplify beam dynamics analysis.

Recently we developed a method to calculate arbitrarily short bunch wakes, including point-charge, from the wake potentials due to a relatively long bunch, thus avoiding the fine mesh requirements in the EM solver. For longitudinal wakes the method is applied as follows [1].

1. Determine analytical singular wake model for a given geometry. For collimator and cavity type structures these are usually given by the optical and diffraction models respectively, see i.e. [2].
2. Determine the length parameter λ_g, which is the first $z\neq0$ location where $W_{\parallel}^{\delta}(z=\lambda_g)$ has a discontinuity or a singularity of its higher derivative with respect to z. The recipe is given in [1] and it is illustrated below.
3. Pick $\sigma_0 \ll \lambda_g$ (but not too short to keep the computation time reasonable) and calculate the wake potential $W_{\parallel}^{\sigma_0}(z)$ with an EM solver.
4. Subtract the singular wake model from the result:
$$D_{\parallel}^{\sigma_0}(z)=W_{\parallel}^{\sigma_0}(z)-W_{\parallel,s}^{\sigma_0}(z). \quad (1)$$
5. Fit the remainder as follows:
$$D_{\parallel}^{\sigma_0}(z\le 3\sigma_0)=\tfrac{\alpha+\beta z}{2}\left(1+\mathrm{erf}\left(\tfrac{z}{\sqrt{2}\sigma_0}\right)\right) + \tfrac{\beta\sigma_0}{\sqrt{2\pi}}\exp\left(-\tfrac{z^2}{2\sigma_0^2}\right), \quad (2)$$
where α and β are fit parameters. Fit range used here, $|z/\sigma_0|\le3$, works very well. However, this choice is rather flexible and may be influenced by practical trade-offs, i.e. fit quality vs. computation time. Maximum z, however, must obey λ_g-$z_{\max}$>few σ_0.
6. For arbitrary short $\sigma\le\sigma_0$, the wake is given by:
$$W_{\parallel}^{\sigma}(z\le 3\sigma_0)=\tfrac{\alpha+\beta z}{2}\left(1+\mathrm{erf}\left(\tfrac{z}{\sqrt{2}\sigma}\right)\right) + \tfrac{\beta\sigma}{\sqrt{2\pi}}\exp\left(-\tfrac{z^2}{2\sigma^2}\right)+W_{\parallel,s}^{\sigma}(z), \quad (3)$$
$$W_{\parallel}^{\sigma}(z>3\sigma_0)=W_{\parallel}^{\sigma_0}(z). \quad (4)$$
7. Similarly, for the point-charge:
$$W_{\parallel}^{\delta}(z\le 3\sigma_0)=(\alpha+\beta z)H(z)+W_{\parallel,s}^{\delta}(z), \quad (5)$$
$$W_{\parallel}^{\delta}(z>3\sigma_0)=W_{\parallel}^{\sigma_0}(z), \quad (6)$$
where $H(z>0)=1$ is the step-function required by causality (z<0 is ahead of the bunch).

Summarizing the steps above, the method basically states that for 0<z< λ_g the point-charge wakefield (with subtracted singular part) is a smooth, slowly varying function. For 0<z<< λ_g it is well approximated by a linear polynomial with coefficients easily found by fitting.

PARAMETER λ_g

λ_g can be determined by a straightforward geometric analysis combined with the causality principle [1]. Figure 1 illustrates how to find λ_g for a simple axially symmetric cavity. In the left pipe, a bunch moving with the speed of light c, carries only static EM fields. As the bunch enters the cavity, the fields get disturbed, exciting the radiation of EM waves. Some wave-fronts (i.e. red or orange)

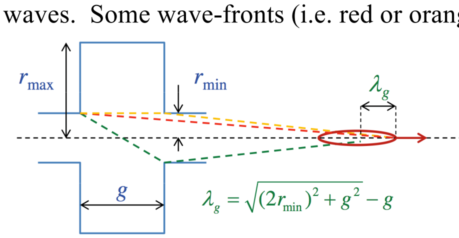

Figure 1: λ_g for a cavity with $r_{\max}/r_{\min}\ge2$ and $g/r_{\min}\ge2^{-1/2}$.

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eventually catch up with the bunch head, thus affecting $W_{\parallel}^{\delta}(z)$ for all values of z. Other wave-fronts (i.e. green) reflect and get delayed, so they can only affect $W_{\parallel}^{\delta}(z \geq d)$, where d is the delay times c.

Each type of reflections has two extreme values, d_e, of the delay, $d_e=d_{min}$ and $d_e=d_{max}$. For instance, all the rays that reflect once off the 2nd cross-sectional step at $r=r_{min}$, are delayed between $d_{min}=0$ (orange ray) and $d_{max}=(4r_{min}^2+g^2)^{1/2}-g$ (green). In general, parameters d_{min} and d_{max} define the z-range where a given reflection type affects $W_{\parallel}^{\delta}(z)$. Thus the points $z=d_e$ contain the discontinuities of the $W_{\parallel}^{\delta}(z)$ function or the singularities of its higher derivatives with respect to z.

The parameter λ_g can thus be found as the minimum of all the non-zero d_e values over all the possible reflection types. For the geometry of Fig. 1, the reflections off the 2nd cross-sectional step have the minimum positive d_e (for the green ray), compared to all the other reflection types, so λ_g is given by the formula shown in the figure.

For shallow cavities, outer-wall reflections result in yet smaller $d_e>0$, causing the replacement, $r_{min}\rightarrow r_{max}-r_{min}$, in the formula for λ_g. Finally, the smallest d_e for a short cavity is due to double reflections between the cavity side-walls, resulting in $\lambda_g=2g$. These three cases include all the possible formulas for λ_g for any r_{max}, r_{min} and g.

For complex geometries the number of different reflection types can be large, but since λ_g is defined by the minimum delay, identifying the relevant type is straightforward [1].

EXTENTION TO TRANSVERSE WAKES

The method above is directly applicable to transverse wakes except for a couple of simple changes. First, since these wakes are not singular, formally step 4 becomes obsolete. Nevertheless, if a short-bunch asymptotic wake model is known, subtracting it and then fitting the residual, $D_{\perp}^{\sigma_0}(z)$, may sometimes result in a better model of the point-charge wake. This will be illustrated later.

Second, the transverse counterpart of (5) should be generally written with an additional parameter κ,

$$W_{\perp}^{\delta}(z \leq 3\sigma_0) = \left(\kappa + \alpha z + \beta z^2\right) H(z), \quad (7)$$

thus modifying (3) and (2) to, respectively, (8) and (9):

$$W_{\perp}^{\sigma}(z \leq 3\sigma_0) = \frac{\kappa+\alpha z+\beta(z^2+\sigma^2)}{2}\left(1+\mathrm{erf}(\tfrac{z}{\sqrt{2}\sigma})\right) + \frac{(\alpha+z\beta)\sigma}{\sqrt{2\pi}}\exp(-\tfrac{z^2}{2\sigma^2}), \quad (8)$$

$$D_{\perp}^{\sigma_0}(z \leq 3\sigma_0) = \frac{\kappa+\alpha z+\beta(z^2+\sigma_0^2)}{2}\left(1+\mathrm{erf}(\tfrac{z}{\sqrt{2}\sigma_0})\right) + \frac{(\alpha+z\beta)\sigma_0}{\sqrt{2\pi}}\exp(-\tfrac{z^2}{2\sigma_0^2}). \quad (9)$$

Note that here, the transverse wake parameters α and β are different from those in the (m=0) longitudinal wake above. Rather, they are related to the corresponding parameters of the m=1 longitudinal wake. Also, in most cases, $W_{\perp}^{\delta}(0)=0$ [3], so the parameter κ is often zero.

Finally, the transverse and the longitudinal wakes for a given structure must have the same parameter λ_g.

EXAMPLE FOR STEP-OUT GEOMETRY

We start by analyzing longitudinal wake potentials due to an axially symmetric step-out structure (Fig. 2, inset). Three wake potentials, computed with time domain EM solver ECHO [4] (used through the rest of this paper), are plotted in Fig. 2 (top).

As σ decreases, the wake potential inside the bunch attains larger negative values ($W_{\parallel}^{\sigma}(z)<0$ corresponds to the energy loss of the particle with longitudinal coordinate z). In the limit $\sigma\rightarrow 0$ this wake potential diverges as σ^{-1} and its singular part is given by the optical model, see i.e. [2],

$$W_{\parallel,s}^{\sigma}(z) = W_{\parallel,opt}^{\sigma}(z) = -\frac{Z_0 c}{(2\pi)^{1/2}\pi\sigma}\ln\frac{r_{max}}{r_{min}} e^{-\frac{z^2}{2\sigma^2}}, \quad (10)$$

$$W_{\parallel,s}^{\delta}(z) = W_{\parallel,opt}^{\delta}(z) = -\frac{Z_0 c}{\pi}\ln\frac{r_{max}}{r_{min}}\delta(z), \quad (11)$$

where Z_0 is the free space impedance.

The same ECHO wake potentials with (10) subtracted, are plotted at the bottom of Fig. 2 (dots). Clearly at $\sigma\rightarrow 0$ these curves approach a well-defined limit shown by the solid black line. This limit function has a discontinuity at z=0, and in the vicinity it can be well approximated by (cf. (5))

$$D_{\parallel}^{\delta}(z) = \left(\alpha+\beta z\right) H(z). \quad (12)$$

Thus, if we determine the parameters α and β, we find the (short-range) point-charge wakefield, which can be written as (5) with the singular part replaced with (11).

For this particular geometry $\lambda_g = 2r_{min} = 2$ cm [1]. Following steps 3-5 above, we pick σ_0=2 mm<<λ_g, calculate $W_{\parallel}^{\sigma_0}(z)$ with ECHO and then determine the parameters α and β by fitting (2). All solid curves in Fig. 2 (bottom) are plotted using these values of α and β.

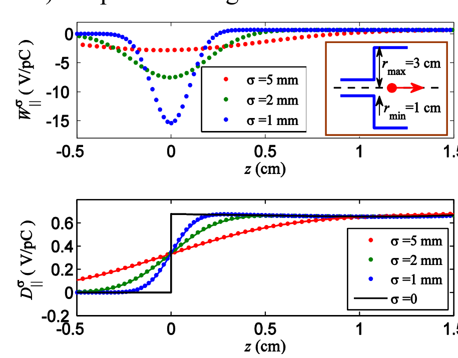

Figure 2: (inset) step-out geometry; (top) ECHO wake potentials; (bottom) same with singular parts subtracted (dots), and (solid) wakes reconstructed from $W_{\parallel,ECHO}^{2mm}$.

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Specifically, the green and blue curves plot (2) with the shown values of σ, while the black curve for the point-charge is the plot of (12). At z>6 mm, all three curves merge together as per (4) and (6). Finally, the red curve, with σ=5mm>σ_0, plots a (numerically performed) convolution of the Gaussian bunch shape with $D_{\|}^{\delta}(z)$ given by the black curve.

Figure 2 illustrates how to find the short-bunch wakes, including the point-charge, from EM solver results due to a relatively long bunch. While even for this simple case no analytical formula exists for $D_{\|}^{\delta}(z)$, the accuracy of this method can be tested further by performing ECHO calculations with shorter and shorter bunch, and observing the results converge to the black curve [1].

Similarly, point-charge wakefield reconstruction can be performed for transverse wakefields, which we will now illustrate for the same geometry of Fig. 2. As stated earlier, transverse wakes are non-singular, so step 4 could be skipped. Then the coefficients α, β and κ are found by fitting the σ_0=2 mm transverse ECHO wake with (9). Substituted in (7)-(8) they define the short-bunch wakes, including the point-charge one. At z>3σ_0 these wakes are given by the transverse analogs of (4) and (6).

Figure 3 shows the wake potentials found in this manner (solid) agreeing very well with the direct calculations by ECHO (dots). In addition, the point-charge wake (black line), agrees well, near z=0, with the optical model, [2],

$$W_{\perp,opt}^{\delta}(z) = -\frac{Z_0 c}{\pi}(r_{\min}^{-2} - r_{\max}^{-2})H(z). \tag{13}$$

In fact, the fitted coefficient κ is within 2×10^{-4} of the value given by (13) (for z>0), which confirms the robustness of our fitting procedure.

Of course, the point-charge wake model found by our method gives a much better approximation away from the origin, where the transverse wake significantly deviates from the step-function behaviour of the optical model.

Alternatively, before the fitting, we could subtract the asymptotic wake due to (13), $W_{\perp,opt}^{2\text{mm}}(z)$. This fit, with κ set to 0, produces very close results to the fit above, i.e. the fitted parameters α are within 0.5% from each other. Therefore, in this example subtracting the asymptotic

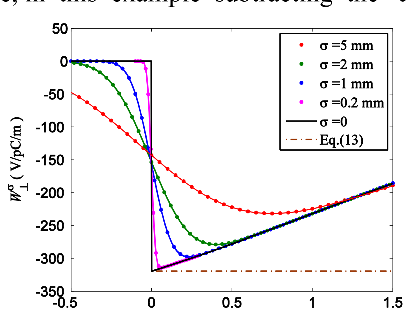

Figure 3: (dots) transverse ECHO wake potentials, and (solid) wakes reconstructed from $W_{\perp,ECHO}^{2\text{mm}}$.

model is optional. Sometimes, however, asymptotic models are known only up to a numerical coefficient. In such cases fitting for the parameter κ is a must.

EXAMPLE FOR CAVITY GEOMETRY

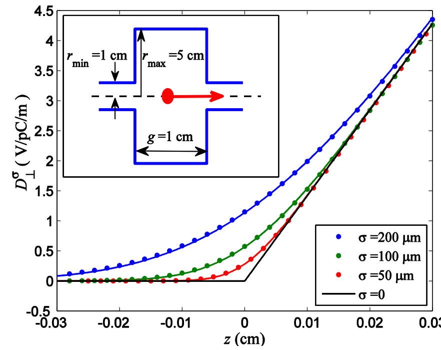

Figure 4: (inset) geometry; (dots) wakes calculated by ECHO and (solid) reconstructed from $W_{\perp,ECHO}^{1\text{mm}}$.

Here we apply the method to an axially symmetric cavity plotted in Fig. 4. Longitudinal wake reconstruction for this structure was described in [1]. For this geometry λ_g≈1.24 cm (see Fig. 1). To proceed with the transverse wake reconstruction, we take σ_0=1mm<<λ_g, and calculate $W_{\perp,ECHO}^{1\text{mm}}$. We then subtract the short-bunch wake asymptotic, $W_{\perp,d}^{1\text{mm}}$, given by the diffraction model, [2],

$$W_{\perp,d}^{\delta}(z) = 2k_d z^{1/2} \quad (z>0), \tag{14}$$

$$W_{\perp,d}^{\sigma}(z) = k_d \sigma^{-1/2} \int f(z/\sigma)\,dz, \tag{15}$$

where $f(s) = e^{-s^2/4}\sqrt{\frac{\pi}{8}|s|}\left(\mathrm{I}_{-1/4}(\frac{s^2}{4}) + \mathrm{sign}(s)\mathrm{I}_{1/4}(\frac{s^2}{4})\right)$, $k_d = -Z_0 c \pi^{-2} r_{\min}^{-3}\sqrt{2g}$, and $\mathrm{I}_{\pm1/4}$ are the Bessel functions.

Unlike the previous example, here subtracting the short-bunch asymptotic substantially improves the final wake model. Indeed, according to (14), the short-bunch wakes are dominated by the $z^{1/2}$ term, which is fitted poorly by a polynomial function.

We proceed by fitting the residual, $D_{\perp,ECHO}^{\sigma_0}(z \le 3\sigma_0)$, to obtain the parameters α and β, and then reconstruct the point-charge wake, (7). With $W_{\perp,d}^{\delta}$ subtracted, it is plotted in Fig. 4 (solid black). Figure 4 also shows that the reconstructed short bunch wakes (solid color) agree perfectly with the direct ECHO calculations (dots).

In conclusion, we extended the point-charge wake reconstruction method of [1] to transverse wakes.

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