An Exact Expression for the Momentum Dependence of the Space Charge Tune Shift in a Gaussian Bunch

Michel Martini
CERN, CH-1211 Geneva 23

Abstract

An analytical approach to calculate the incoherent tune shift in the presence of non-linear space charge forces is described in this paper. Closed form expressions to evaluate the dependence of the space charge detuning on betatron amplitude have been derived previously for a beam of elliptic cross section with Gaussian distribution in transverse dimensions, under the condition that each particle has the same momentum. The present computation considers in addition the dependence of the space charge detuning on momentum, for a beam with Gaussian momentum distribution. Two effects are taken into account for an exact calculation: the widening of the beam due to momentum spread and the variation of the tune shift with the longitudinal position in the bunch. Application to the present high intensity beams in the CERN PS machine and to the foreseen beam for LHC is discussed.

I. INTRODUCTION

The understanding of the space charge effects is of primary importance for low energy circular accelerators and storage rings especially for beams of high intensity and brightness as required for the LHC [1]. Theoretical models which permit reliable numerical simulation are desirable to analyze the beam behaviour. An analytical approach to calculate the incoherent tune shifts in circular accelerators and storage rings in the presence of nonlinear space charge forces is described hereinafter. The model takes into account the dependence of the space-charge induced detuning on betatron amplitude and momentum, for a beam with Gaussian distributions in transverse dimensions and momenta.

II. EQUATIONS OF MOTION

In a linear lattice the synchro-betatron motion for a charged particle in the presence of space charge forces can be derived from the Hamiltonian [2]

$$
\begin{aligned}
H(x,z,\sigma,p_x,p_z,p_\sigma) &= \frac{eU(x,z,\sigma)}{m_0c^2\beta^2\gamma} + \frac{p_\sigma^2}{2\gamma^2} \\
&- (K_x x + K_z z)\,p_\sigma + \frac{1}{2}\left(g_x x^2 + g_z z^2 + p_x^2 + p_z^2\right) \\
&+ \frac{L}{2\pi k}\frac{eV(s)}{m_0c^2\beta^2}\cos\left(\frac{2\pi h}{L}\sigma + \varphi\right) \qquad (1)
\end{aligned}
$$

in which x, p_x, z, p_z, $\sigma \overset{def}{=} s-\beta ct$, $p_\sigma \overset{def}{=} \Delta E/\beta^2 E$ describe the synchro-betatron oscillations, s is the arc length, E is the total energy of the particle and β, γ the relativistic parameters. $U(x,z,\sigma)$ is the potential generated by the beam, $g_{x,z}$ is the quadrupole strength, $K_{x,z}$ the curvature of the reference orbit, L its length, $V(s)$ and φ are the accelerating field and the phase, h the harmonic number, m_0 and e the rest mass and the charge of the particle.

The off-momentum trajectory may be placed at the center of phase space by means of a canonical transformation with generating function [2]

$$
\begin{aligned}
F_2(x,z,\sigma,\tilde{p}_x,\tilde{p}_z,\tilde{p}_\sigma) &= \\
&\tilde{p}_x(x - D_x\Delta) + \tilde{p}_z(z - D_z\Delta) + (D_x' x + D_z' z)\Delta \\
&-(D_x' D_x + D_z' D_z)\frac{\Delta^2}{2} + \tilde{p}_\sigma\sigma \qquad (2)
\end{aligned}
$$

where $\Delta \overset{def}{=} \Delta p/p$ is the momentum deviation, $D_{x,y}$ is the dispersion function and a prime implies differentiation with respect to s. The transformation equations lead to the known expressions

$$\tilde{y} = y - D_y\Delta \qquad \tilde{p}_y = p_y - D_y'\Delta \qquad (3)$$

$$\tilde{\sigma} = \sigma + \left(1 - \frac{\tilde{p}_\sigma}{\gamma^2}\right)(-D_x\tilde{p}_x - D_z\tilde{p}_z + D_x'\tilde{x} + D_z'\tilde{z}) \qquad (4)$$

and $\tilde{p}_\sigma = p_\sigma$, where y stands either for x or z. Assuming that there is no dispersion in the cavities ($V(s)D_y = 0$, $V(s)D_y' = 0$) and using the "oscillator model" for the synchrotron motion [2], with expansion of the cosine term to the second order, the Hamiltonian reduces to

$$
\begin{aligned}
\tilde{H}(\tilde{x},\tilde{z},\tilde{\sigma},\tilde{p}_x,\tilde{p}_z,\tilde{p}_\sigma) &= \frac{eU(\tilde{x},\tilde{z},\tilde{\sigma})}{m_0c^2\beta^2\gamma} + \frac{\eta}{2}\tilde{p}_\sigma^2 \\
&-\frac{1}{2}\left(g_x\tilde{x}^2 + g_z\tilde{z}^2 + \tilde{p}_x^2 + \tilde{p}_z^2\right) + \frac{\Omega_\sigma^2}{2\eta\beta^2c^2}\tilde{\sigma}^2 \qquad (5)
\end{aligned}
$$

where $\eta = 1/\gamma^2 - \alpha_p$ is the phase slip factor, with the momentum compaction factor and the synchrotron frequency

$$\alpha_p = \frac{1}{L}\int_{s_0}^{s_0+L}(K_x D_x + K_z D_z)\,ds \qquad (6)$$

$$\Omega_\sigma^2 = \frac{2\pi h\eta c^2\cos\varphi}{EL^2}\int_{s_0}^{s_0+L} eV(s)ds \qquad (7)$$

We introduce the action-angle variables $I_{x,z,\sigma}$, $\psi_{x,z,\sigma}$ and $\tilde{I}_{x,z,\sigma}$, $\tilde{\psi}_{x,z,\sigma}$ use two consecutive canonical transformations, transforming first $\tilde{H}$ into K and then K into $\tilde{K}$,

0-7803-1203-1/93$03.00 © 1993 IEEE

PAC 1993

with generating functions [2]

$$F_1(\tilde{x},\tilde{z},\tilde{\sigma},\psi_x,\psi_z,\psi_\sigma) = -\frac{\tilde{\sigma}^2}{2\beta_\sigma}(\alpha_\sigma+\tan\psi_\sigma)$$
$$-\frac{\tilde{x}^2}{2\beta_x}(\alpha_x+\tan\psi_x)-\frac{\tilde{z}^2}{2\beta_z}(\alpha_z+\tan\psi_z) \quad (8)$$

$$F_2(\tilde{I}_x,\tilde{I}_z,\tilde{I}_\sigma,\psi_x,\psi_z,\psi_\sigma,s) =$$
$$\tilde{I}_x\left(\psi_x+\frac{Q_x s}{R}-\int_0^s\frac{ds}{\beta_x}\right)+\tilde{I}_z\left(\psi_z+\frac{Q_z s}{R}-\int_0^s\frac{ds}{\beta_z}\right)$$
$$+\tilde{I}_\sigma\left(\psi_\sigma+\frac{Q_\sigma s}{\eta R}-\int_0^s\frac{\eta}{\beta_\sigma}ds\right) \quad (9)$$

Then the equations of motion can be written as

$$\tilde{y}=\sqrt{2I_y\beta_y}\cos\psi_y \qquad \tilde{p}_y=-\sqrt{\frac{2I_y}{\beta_y}}(\alpha_y\cos\psi_y+\sin\psi_y) \quad (10)$$

with

$$\tilde{I}_y = I_y \qquad \tilde{\psi}_y = \psi_y+\frac{Q_y s}{R}-\int_0^s\frac{F_y}{\beta_y}ds \quad (11)$$

in which $R=L/2\pi$, y now stands either for x, z, or σ, and $F_x=F_z=1$, $F_\sigma=\eta$. The Hamiltonian $\tilde{K}$ after substituting the transformed variables becomes

$$\tilde{K}(\tilde{I}_x,\tilde{I}_z,\tilde{I}_\sigma,\tilde{\psi}_x,\tilde{\psi}_z,\tilde{\psi}_\sigma) = \frac{\tilde{I}_x Q_x}{R}+\frac{\tilde{I}_z Q_z}{R}$$
$$+\frac{\tilde{I}_\sigma Q_\sigma}{R}+\frac{e}{m_0c^2\beta^2\gamma}U(\tilde{I}_x,\tilde{I}_z,\tilde{I}_\sigma,\tilde{\psi}_x,\tilde{\psi}_z,\tilde{\psi}_\sigma) \quad (12)$$

Using perturbation theory a final canonical transformation to new action-angle variables $\bar{I}_{x,z,\sigma}$ and $\bar{\psi}_{x,z,\sigma}$ may be further applied so that to first order the transformed Hamiltonian $\bar{K}$ is the average of the old Hamiltonian $\tilde{K}$ over the old angle variables [2]. Moreover, assuming that the disturbing potential is small, the new action variables may be replaced by the old ones so that the transformation equation for the angle variable is

$$\bar{\psi}_y' = \frac{\partial\bar{K}}{\partial I_y} = \frac{Q_y}{R}+\frac{eF_y}{m_0c^2\beta^2\gamma}\frac{\partial\langle U\rangle}{\partial I_y} \quad (13)$$

where the average is taken over both the $\tilde{\psi}_{x,z,\sigma}$ and s.

III. SPACE CHARGE POTENTIAL

Bunched beams of ellipsoidal shape with half-dimensions a, b, c defined as $\sqrt{2}$ times the r.m.s. beam sizes σ_x, σ_z, σ_σ, and with Gaussian charge density in the ellipsoid

$$\varrho(x,z,\sigma) = \frac{N_b e}{\pi^{3/2}abc}\exp\left(-\frac{x^2}{a^2}-\frac{z^2}{b^2}-\frac{\sigma^2}{c^2}\right) \quad (14)$$

yield non-linear space charge forces generated by the scalar potential [3]

$$U(x,z,\sigma) = -\frac{N_b e}{4\pi^{3/2}\epsilon_0\gamma^2}$$
$$\times\int_0^\infty\frac{1-\exp\left(-\frac{x^2}{a^2+t}-\frac{z^2}{b^2+t}-\frac{\gamma^2\sigma^2}{\gamma^2c^2+t}\right)}{\sqrt{(a^2+t)(b^2+t)(\gamma^2c^c+t)}}dt \quad (15)$$

where N_b is the number of particles per bunch and ϵ_0 the dielectric constant of vacuum. The effects of image fields due to the conducting vacuum pipe have been ignored.

Combining Eqs. 3-4 and Eqs. 10-11, and considering the "smooth approximation" $\beta_y \approx R/Q_y$ and $\beta_\sigma \approx \eta R/Q_\sigma$ gives, replacing $\tilde{\psi}_{x,z,\sigma}$ by $\bar{\psi}_{x,z,\sigma}$ to first approximation

$$y = \hat{y}\cos\psi_y - \bar{D}_y\hat{\Delta}\sin\psi_y \quad (16)$$

$$\sigma = \hat{\sigma}\cos\psi_\sigma - \bar{D}_xQ_x\hat{x}\sin\psi_x - \bar{D}_zQ_z\hat{z}\sin\psi_z \quad (17)$$

in which $\hat{y}=\sqrt{2I_yR/Q_y}$, $\hat{\sigma}=\sqrt{2I_\sigma Q_\sigma/\eta R}$, $\hat{\Delta}=Q_\sigma\hat{\sigma}/\eta R$ denote the amplitudes of the synchro-betatron oscillations, $\bar{D}_y$ is the mean dispersion function. Then, expanding $U(x,z,\sigma)$ in series and using the above expressions yields

$$\langle U\rangle = \frac{N_b e}{4\pi^{3/2}\epsilon_0\gamma^2}\left\langle\sum_{n=1}^\infty\frac{(-1)^n}{n!}\sum_{j_1+j_2+j_3=n}J(j_1,j_2,j_3)\right.$$
$$\times\left(\frac{\hat{x}\cos\psi_x-\bar{D}_x\hat{\Delta}\sin\psi_\sigma}{a}\right)^{2j_1}\left(\frac{\hat{z}\cos\psi_z-\bar{D}_z\hat{\Delta}\sin\psi_\sigma}{b}\right)^{2j_2}$$
$$\times\left.\left(\frac{R\hat{\sigma}\cos\psi_\sigma-\bar{D}_xQ_x\hat{x}\sin\psi_x-\bar{D}_zQ_z\hat{z}\sin\psi_z}{Rc}\right)^{2j_3}\right\rangle \quad (18)$$

with the elliptic integral

$$J(j_1,j_2,j_3) =$$
$$\int_0^\infty\frac{dt}{(1+\frac{bt}{a})^{j_1+\frac{1}{2}}(1+\frac{at}{b})^{j_2+\frac{1}{2}}(1+\frac{ab}{\gamma^2c^2}t)^{j_3+\frac{1}{2}}} \quad (19)$$

When the longitudinal dimension of the bunch is much larger than the transverse dimensions, Eq. 19 may be approximately evaluated using the recursion formulae [4]

$$J(0,0,n) \approx \ln\left(\frac{4\gamma c}{a+b}\right)^2 - 2\sum_{i=1}^n\frac{1}{2i-1} \quad (20)$$

$$J(1,0,j_3) \approx \frac{2}{\alpha+1} \quad (21)$$

$$J(j_1,0,j_3) \approx \frac{\alpha-(n-1)J(j_1-1,0,j_3)}{(n-1/2)(\alpha^2-1)} \quad (22)$$

$$J(j_1,j_2,j_3) \approx \frac{\alpha-\alpha^2(j_1+1/2)J(j_1+1,j_2-1,j_3)}{j_2-1/2} \quad (23)$$

where $\alpha=b/a$. Eqs. 21-23 are independent on j_3.

IV. TUNE SHIFT FORMULAE

Integrating Eq. 13 through one machine turn gives an additional phase advance which is identified with the space charge detuning

$$\Delta Q_y = \frac{eRF_y}{m_0c^2\beta^2\gamma Q_y\hat{y}}\frac{\partial\langle U\rangle}{\partial\hat{y}} \quad (24)$$

Assuming that the machine tune is removed from non-linear and coupling resonances Eq. 18 may be averaged individually over the phase advances. The resulting calculations yield a potential which depends only on the synchro-betatron amplitudes $\hat{x}$, $\hat{z}$, and $\hat{\Delta}$. Hence, differentiating this potential and inserting the result into Eq. 24 gives the betatron and momentum amplitude dependence of the incoherent tune shift. Thus, the horizontal space charge tune shift is

$$\Delta Q_x(\hat{x},\hat{z},\hat{\Delta}) = -\Delta Q_{0,x}\left(1+\frac{b}{a}\right)$$
$$\times \sum_{n=1}^{\infty}\frac{(-1)^n}{2^{2n}}\sum_{j_1=0}^{n}\sum_{j_2=0}^{n-j_1}\frac{(2j_1)!(2j_2)!(2j_3)!}{j_1!j_2!j_3!}J(j_1,j_2,j_3)$$
$$\times \sum_{i=0}^{j_1}\sum_{k=0}^{j_2}\sum_{l=0}^{j_3}\sum_{m=0}^{j_3-l}\frac{(2(j_1-i)+2(j_2-k))!}{(2(j_1-i))!(2(j_2-k))!}\frac{1}{i!k!l!m!}$$
$$\times \frac{i+m}{(i+m)!(j_3-l-m)!(j_3+k-l-m)!(j_1+j_2-i-k)!}$$
$$\times \frac{1}{(j_1+j_2-i-k+l)!}\left(\frac{\hat{x}}{a}\right)^{2(i+m-1)}\left(\frac{\hat{z}}{b}\right)^{2(j_3+k-l-m)}$$
$$\times \left(\frac{\hat{\Delta}}{\delta}\right)^{2(j_1+j_2+k-l-m)}\left(\frac{\bar{D}_xQ_xa}{Rc}\right)^{2m}\left(\frac{\bar{D}_zQ_zb}{Rc}\right)^{2(j_3-l-m)}$$
$$\times \left(\frac{\bar{D}_xQ_\sigma c}{\eta Ra}\right)^{2(j_1-i)}\left(\frac{\bar{D}_zQ_\sigma c}{\eta Rb}\right)^{2(j_2-k)} \qquad (25)$$

in which $\delta = Q_\sigma c/\eta R$ and $\Delta Q_{0,x}$ is the Laslett tune shift in the center of a Gaussian beam

$$\Delta Q_{0,x} = -\frac{k_b N_b R r_0}{\pi B_f Q_x \beta^2\gamma^3}\frac{1}{a(a+b)} \qquad (26)$$

where $B_f = k_b\sigma_\sigma/\sqrt{2\pi}R$ is the bunching factor defined as the ratio of the mean to peak line charge density, k_b being the number of bunches and $r_0 = e^2/4\pi\epsilon_0 m_0 c^2$ the classical particle radius. Expressions for the vertical and the longitudinal tune shifts can be written similarly. Eq. 25 reduces to the Keil formula when $c \gg a, b$ and $\hat{\Delta}=0$ [4].

V. RESULTS AND CONCLUSION

The foregoing formulae have been applied at the 1 GeV injection into the PS to the present high intensity beam delivered to the SPS, and at the 1.4 GeV injection into the PS to the high brilliance beam required for the LHC [1]

- SPS: 20 bunches of 10^{12} protons, 1 GeV, 55 ns long, 1σ-momentum spread $\sigma_\Delta = 0.75\times10^{-3}$, 1σ-normalized emittances $\epsilon^*_{x,z} = 12.5, 6.25\ \mu m$[1].
- LHC: 8 bunches of 1.75×10^{12} protons, 1.4 GeV, 190 ns long, 1σ-momentum spread $\sigma_\Delta = 1.25\times10^{-3}$, 1σ-normalized emittances $\epsilon^*_{x,z} = 3.5, 1.75\ \mu m$.

Figures 1-2 show the amplitude dependence of the space charge tune shifts for these two beams. The nominal tunes are $Q_{x,z}$=6.22, 6.28. Calculations have been performed for amplitudes varying between 0 to $2\sigma_{x,z,\Delta}$ with the series expansions pushed to the 15th order. The Laslett tune shifts are $\Delta Q^{\mathrm{SPS}}_{x,z} = -0.21, -0.30$ and $\Delta Q^{\mathrm{LHC}}_{x,z} = -0.18, -0.31$.

[1] The beam half-dimensions are $a = \sqrt{2}\sqrt{\epsilon^*_x R/\beta\gamma Q_x + \bar{D}_x^2\sigma_\Delta^2}$, $b = \sqrt{2}\sqrt{\epsilon^*_z R/\beta\gamma Q_z + \bar{D}_z^2\sigma_\Delta^2}$ and $c = \sqrt{2}\eta R\sigma_\Delta/Q_\sigma$, $Q_\sigma = \Omega_\sigma R/\beta c$.

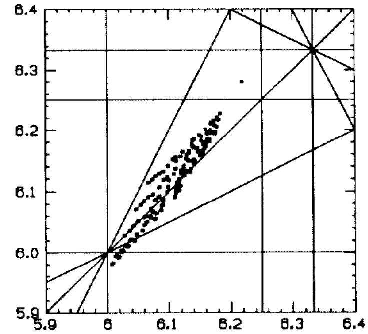

Figure 1: Tune diagram at the 1 GeV PS injection for the present beam for SPS.

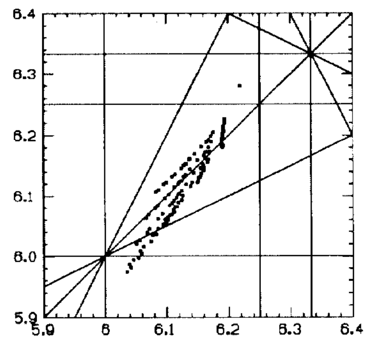

Figure 2: Tune diagram at the 1.4 GeV PS injection for the future beam for LHC.

The betatron and momentum amplitudes having Rayleigh distributions [4], the mean tune shifts have been computed from Eq. 15 by averaging Eq. 24 over all the amplitudes using Monte-Carlo integration. The results are $\langle\Delta Q\rangle^{\mathrm{SPS}}_{x,z}$=−0.09, −0.13 and $\langle\Delta Q\rangle^{\mathrm{LHC}}_{x,z}$=−0.08, −0.13. Thus, the weighted average of the detuning is less than half the Laslett tune shift.

The tune diagrams Q_x-Q_z for the LHC and the SPS beams look almost similar. Thus, by raising the PS input energy to 1.4 GeV the tune spreads of the LHC beam will be maintained at the present limiting level.

VI. REFERENCES

[1] R. Cappi, R. Garoby, S. Hancock, M. Martini, N. Rasmussen, J.P. Riunaud, K. Schindl, H. Schönauer, "The CERN PS Complex as part of the LHC injector chain", Proc. 1991 IEEE PAC, p. 171.

[2] D.P. Barber, H. Mais, G. Ripken, F. Willeke, "Nonlinear theory of coupled synchro-betatron motion", DESY report 86-147 (1986).

[3] K. Takayama, "A new method for the potential of a 3-dimensional nonuniform charge distribution", Lettere al Nuovo Cimento, Vol. 34, p. 190 (1980).

[4] E. Keil, "Non-linear space charge effects I", CERN report ISR-TH/72-7 (1972).