Evolution of Hadron Beams under Intrabeam Scattering*

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Abstract

Based on assumptions applicable to many circular accelerators, we simplify into analytical forms the growth rates of hadron beams under Coulomb intrabeam scattering (IBS). Because of the dispersion that correlates the horizontal closed orbit to the momentum, the scaling behavior of the growth rates are drastically different at energies low and high compared with the transition energy. At high energies, the rates are approximately independent of the energy. Asymptotically, the horizontal and longitudinal beam amplitudes are linearly related by the average dispersion. At low energies, the beam evolves such that the velocity distribution in the rest frame becomes isotropic in all the directions.

1 INTRODUCTION

During the last decade, many theories have been developed on the subject of intrabeam Coulomb scattering[1-4] of the hadron beam. These theories assume that the particle distribution remains Gaussian in the six-dimensional phase space. The rates of growth in the rms beam amplitudes are expressed in complex integral forms.

This paper attempts to describe the principle scaling behavior of the growth rates and, based upon which, the evolution of the beam at different energy regimes. In section 2, the previous expressions for the rates are simplified into analytical forms, provided that the lattice of the accelerator mainly consists of regular cells. The dependence of the growth rates on the beam charge state, mass, energy, phase-space area, and the machine transition energy is obtained. Based on these formulae, we derive the scaling laws of the beam evolution in different dimensions under various circumstances. In particular, the results for the high-energy and low-energy regimes are discussed in Sections 3 and 4, respectively.

2 BEAM GROWTH RATES

The growth of the particle beam under intrabeam scattering is usually described by the relative time derivatives of the rms horizontal betatron amplitude σ_x, vertical amplitude σ_y, and fractional momentum deviation σ_p, respectively. Assume that the scatterings mostly occur at small scattering angles, and that the particle distribution remains Gaussian in six-dimensional phase space. When the transverse motions are not coupled, these rates are obtained at any location of the machine[4]

$$\begin{bmatrix} \frac{1}{\sigma_p}\frac{d\sigma_p}{dt} \\ \frac{1}{\sigma_x}\frac{d\sigma_x}{dt} \\ \frac{1}{\sigma_y}\frac{d\sigma_y}{dt} \end{bmatrix} = \frac{A_0}{2}\int e^{-Dz}\ln(1+C^4z^2)\begin{bmatrix} n_b(1-d^2)g_1 \\ a^2g_2+(d^2+\overline{d}^2)g_1 \\ b^2g_3 \end{bmatrix} \tag{1}$$

$$\times \sin\theta d\theta d\phi dz$$

where $A_0 = \dfrac{cr_0^2NZ^4\beta_x\beta_y}{32\pi^2A^2\sigma_x^2\sigma_y^2\sigma_p\sigma_s\beta^3\gamma^4}$, $r_0 = \dfrac{e^2}{m_0c^2}$,

$$d = \frac{D_p\sigma_p}{(\sigma_x^2+D_p^2\sigma_p^2)^{1/2}},\ \overline{d} = \frac{\overline{D}_pd}{D_p},\ \overline{D}_p = \alpha_xD_p+\beta_xD_p',$$

$$a = \frac{\beta_xd}{D_p\gamma},\ b = \frac{\beta_y\sigma_x}{\beta_x\sigma_y}a,$$

and

$$D = \cos^2\theta + b^2\sin^2\theta\sin^2\phi + (a\sin\theta\cos\phi - \overline{d}\cos\theta)^2,$$

$$C = 2\beta\sigma_p\left[\sigma_y(1-d^2)/r_0\right]^{1/2},$$

$$\begin{aligned} g_1 &= 1-3\cos^2\theta,\\ g_2 &= \cos^2\theta - 2\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \\ &\quad +6\overline{d}\cos\theta\sin\theta\cos\phi/a,\\ g_3 &= \cos^2\theta + \sin^2\theta\cos^2\phi - 2\sin^2\theta\sin^2\phi. \end{aligned}$$

Here, the prime denotes the derivative with respect to the azimuthal displacement, D_p is the horizontal dispersion, $\alpha_{x,y}$ and $\beta_{x,y}$ are the Courant-Snyder parameters, γ is the Lorentz factor, n_b is equal to 1 if the beam is azimuthally bunched, and is equal to 2 if it is not. For bunched beams, σ_s is the rms bunch length and N is the number of particles per bunch; for un-bunched beams, N is the total number of particles and $\sigma_s = L/2\sqrt{\pi}$, where L is the circumference of the machine. The quantity $d < 1$ is the effective ratio between the longitudinal and horizontal total amplitude. The actual growth rates observed over a time long compared with the revolution period, are calculated by averaging Eq. 1 over the circumference. This averaging process is implicitly implied in almost all the following equations.

Eq. 1 can in many cases be simplified into analytical forms. Firstly, the quantity $\ln(1+C^4z^2)$ in Eq. 1 has a weak dependence on the beam configuration. It can be substituted[2] by a constant $2L_c$, where L_c is about 20. With this simplification, the integration over z can be performed. Secondly, we assume that the accelerator consists mostly of regular cells, so that the variation in $D_p/\beta_x^{1/2}$ is small along the circumference. Terms including $\overline{D_p}$ and $\overline{d}$

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in Eq. 1 are thus negligible. Replacing $\sin^2\phi$ and $\cos^2\phi$ with their average value 1/2, Eq. 1 is simplified by integrations over z, θ and ϕ

$$\begin{bmatrix} \frac{1}{\sigma_p}\frac{d\sigma_p}{dt} \\ \frac{1}{\sigma_x}\frac{d\sigma_x}{dt} \\ \frac{1}{\sigma_y}\frac{d\sigma_y}{dt} \end{bmatrix} = 4\pi A_0 L_c F(\chi) \begin{bmatrix} n_b(1-d^2) \\ -a^2/2+d^2 \\ -b^2/2 \end{bmatrix}, \tag{2}$$

where

$$\chi = (a^2+b^2)/2 \geq 0. \tag{3}$$

As shown in Fig. 1, the function

$$F(\chi) = \frac{-3+(1+2\chi)I(\chi)}{1-\chi} \tag{4}$$

with

$$I(\chi) = \begin{cases} \frac{1}{\sqrt{\chi(\chi-1)}}\mathrm{Arth}\sqrt{\frac{\chi-1}{\chi}} & \chi \geq 1; \\ \frac{1}{\sqrt{\chi(1-\chi)}}\arctan\sqrt{\frac{1-\chi}{\chi}} & \chi < 1 \end{cases} \tag{5}$$

is a smooth function of χ. It is positive when $\chi < 1$, zero when $\chi = 1$, and negative when $\chi > 1$. $F(\chi)$ has the asymptotic expression

$$F(\chi) = \begin{cases} \frac{\pi}{2\sqrt{\chi}} & \chi \ll 1; \\ -\frac{\ln\chi}{\chi} & \chi \gg 1. \end{cases} \tag{6}$$

In terms of the normalized transverse emittance $\epsilon_{x,y} = \beta\gamma\sigma_{x,y}^2/\beta_{x,y}$ and longitudinal bunch area $S = \pi m_0 c^2\gamma\sigma_s\sigma_p/\beta^3 cA$ in phase space, Eq. 2 can be rewritten

$$\begin{bmatrix} \frac{1}{\sigma_p}\frac{d\sigma_p}{dt} \\ \frac{1}{\sigma_x}\frac{d\sigma_x}{dt} \\ \frac{1}{\sigma_y}\frac{d\sigma_y}{dt} \end{bmatrix} = \frac{Z^4N}{A^2}\frac{r_0^2 m_0 c^2 L_c}{8\beta^4\gamma\epsilon_x\epsilon_y S}F(\chi) \begin{bmatrix} n_b(1-d^2) \\ -a^2/2+d^2 \\ -b^2/2 \end{bmatrix}. \tag{7}$$

Except for the form factors χ, d, a, and b that depend on the ratio of the beam amplitudes in different dimension, the rates are linearly proportional to the density in the six-dimensional phase space, and are strongly dependent ($\sim Z^4/A^2$) on the charge state of the particle.

The growths in the longitudinal and vertical amplitudes are both caused by the variation of the velocitie in the corresponding direction. The growth in the horizontal amplitude, on the other hand, is caused partly by the variation in the horizontal velocity, and partly by the change in the betatron closed orbit when the momentum of the particle is varied during the collision. It can be easily verified that the first (or second) part dominates when the beam is below (or above) the transition energy of the machine.

The coupling between the horizontal and vertical motion averages the growth rates in the transverse dimension. If the motion is fully coupled[4] within time periods much shorter than the IBS diffusion time, the average rates become

$$\begin{bmatrix} \frac{1}{\sigma_p}\frac{d\sigma_p}{dt} \\ \frac{1}{\sigma_{x,y}}\frac{d\sigma_{x,y}}{dt} \end{bmatrix} = \frac{Z^4N}{A^2}\frac{r_0^2 m_0 c^2 L_c}{8\beta^4\gamma\epsilon_x\epsilon_y S}F(\chi) \begin{bmatrix} n_b(1-d^2) \\ (-\chi+d^2)/2 \end{bmatrix}. \tag{8}$$

3 BEAM EVOLUTION AT HIGH ENERGIES

In a typically circular accelerator, the transition energy γ_T is approximately equal to the average value of β_x/D_p in the regular cells. When the beam energy is high $\gamma \gg \gamma_T$, the growth in horizontal direction results mostly from the variation of the betatron closed orbit during the exchange of the particle momentum ($a^2 \ll d^2$). The growths in horizontal and longitudinal amplitudes are therefore proportional to each other (Eq. 2).

Consider the case that the vertical σ_y is very small, i.e. on the average

$$\frac{\sigma_y}{\sigma_y} < \frac{d}{2}\frac{\gamma_T}{\gamma}, \qquad \gamma \gg \gamma_T. \tag{9}$$

It may be verified that $\chi > 1$, and $F(\chi) < 0$. According to Eq. 2, both the horizontal and longitudinal amplitudes shrink, while the vertical one grows. The beam evolves until Eq. 9 is no longer satisfied.

When the vertical amplitude is no longer small so that $\chi < 1$, both horizontal and longitudinal amplitudes grow.[4] Consider the effective ratio between the horizontal betatron amplitude and longitudinal amplitude, $C_H \equiv n_b n_c \sigma_x^2/D_p^2\sigma_p^2$ where n_c is equal to 1 if the horizontal and vertical motions are not coupled , and is equal to 2 if they are fully coupled. Using Eq. 2, the rate of change of C_H can be derived on the average

$$\frac{dC_H}{dt} = 4\pi A_0 L_c d^2 C_H F(\chi)(1-C_H). \tag{10}$$

This rate is positive if C_H is less than 1, and is negative if C_H is larger than 1. Therefore, σ_x and σ_p grow such that asymptotically the quantity C_H approaches 1, or

$$n_b n_c \sigma_x^2 \approx D_p^2\sigma_p^2, \qquad \gamma \gg \gamma_T. \tag{11}$$

σ_x and σ_p are related only by the average dispersion D_p.

In a typical storage ring like the Relativistic Heavy Ion Collider (RHIC), the beams are stored at energies much higher than the transition energy. Due to coupling and injection conditions, the horizontal and vertical betatron amplitudes are about the same. The growth rates can be explicitly written from Eq. 2 by using Eq. 6

$$\begin{bmatrix} \frac{1}{\sigma_p}\frac{d\sigma_p}{dt} \\ \frac{1}{\sigma_x}\frac{d\sigma_x}{dt} \end{bmatrix} = \frac{Z^4N}{A^2}\frac{\pi r_0^2 m_0 c^2 L_c}{16\beta^4\gamma_T\epsilon_x\epsilon_y S} \begin{bmatrix} n_b(1-d^2)/d \\ d/n_c \end{bmatrix}. \tag{12}$$

Their dependence on the energy of the beam, which appears only in the form factor d, is usually weak. After the initial stage of storage, the asymptotic configuration Eq. 11 will be approximately reached ($d \approx n_b n_c/(1+n_b n_c)$).

4 BEAM EVOLUTION AT LOW ENERGIES

Beam evolution at energies much lower than the transition energy of the machine can be studied similarly. At low energies, $a^2 \gg d^2$, the growth in horizontal amplitude is mostly caused by the variation in the horizontal velocity alone. Eq. 2 indicates that the growths in horizontal and vertical amplitudes are proportional to each other.

Consider the case that the longitudinal σ_p is very small, i.e. on the average

$$\frac{D_p \sigma_p}{\sigma_x} < \frac{\gamma}{\gamma_T}\sqrt{\frac{2}{1+C_L}}, \qquad \gamma \ll \gamma_T, \tag{13}$$

where $C_L \equiv \beta_y^2 \sigma_x^2 / \beta_x^2 \sigma_y^2$ is the betatron velocity ratio between horizontal and vertical directions. It may be verified that $\chi < 1$, and $F(\chi) > 0$. According to Eq. 2, both horizontal and vertical amplitudes shrink, while the longitudinal one grows. The beam evolves until Eq. 13 is no longer satisfied.

When the longitudinal amplitude is no longer small so that $\chi > 1$, both horizontal and vertical amplitudes grow. Using Eq. 2, the rate of change of C_L can be derived

$$\frac{dC_L}{dt} = -4\pi A_0 L_c a^2 C_L F(\chi)(1-C_L). \tag{14}$$

σ_x and σ_y grow in such a way that asymptotically the quantity C_L approaches 1. Combining with Eq. 13 and the previous results, we therefore obtain the asymptotic beam configuration at low energies

$$\frac{\sigma_x}{\beta_x} \approx \frac{\sigma_y}{\beta_y} \approx \frac{\sigma_p}{\gamma}, \qquad \gamma \ll \gamma_T. \tag{15}$$

The three quantities in Eq. 15 are proportional to the horizontal, vertical, and longitudinal velocities in the rest frame of the particles, respectively. Eq. 15 implies that the beam evolves such that the velocity distribution in the rest frame becomes isotropic in all the three directions.

5 CONCLUSIONS AND DISCUSSIONS

Based on assumptions applicable to many circular accelerators, we simplified the general integral expressions (Eq. 1) of the IBS growth rates into analytical forms (Eq. 2). The rates are expressed in terms of the beam charge state, mass, energy, phase-space area, and the machine transition energy, both for the un-coupled (Eq. 7) and fully coupled (Eq. 8) cases. They have been shown to be linearly proportional to the particle density in the six-dimensional phase space. Because of the dispersion that correlates the horizontal closed orbit to the momentum, the effect of intrabeam scattering is different at different energy regime. At energies much higher than the transition energy, the growth rates have been shown to be approximately independent of the energy (Eq. 12).

Quantitative comparisons have been performed on the average growth rates between the simple estimate (Eq. 7) and the detailed evaluation (Eq. 1) including[4] lattice variation using the actual RHIC lattice. For both the injection (low energy) and storage (high energy) cases, the relative deviation between them is about 20%.

The evolution of the beam in different dimensions has been investigated at energies both much higher and much lower than the transition energy. At high energies, the asymptotic beam amplitudes in horizontal and longitudinal direction are shown to be linearly related by the average dispersion (Eq. 11). At low energies, on the other hand, the beam evolves such that the velocity distribution in the rest frame becomes isotropic in horizontal, vertical, and longitudinal directions (Eq. 15).

During the entire analysis it has been assumed that the beam distribution remains Gaussian in the phase space. This assumption is valid only when the beam amplitudes are small compared with the machine aperture. In the case that beam loss occurs due to aperture limitation, different approaches[5] have to be adopted.

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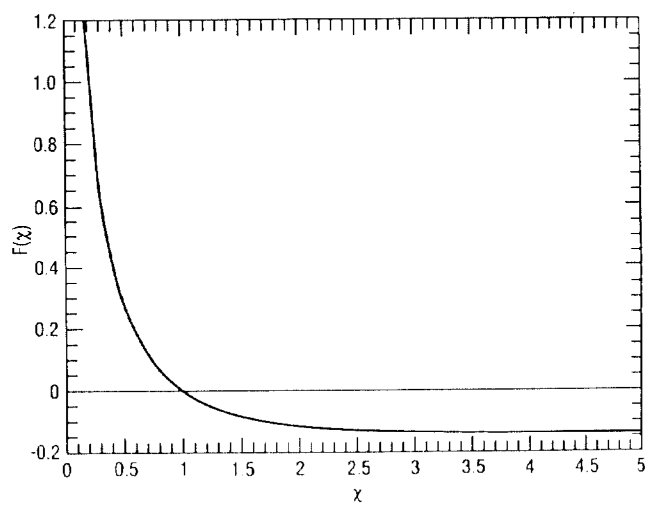

Fig. 1. Function $F(\chi)$ with $0 \le \chi < \infty$.