Confinement and Stability of a Crystal Beam*

Alessandro G. Ruggiero
Brookhaven National Laboratory
Upton, Long Island, New York 11973, USA

Abstract

The following is an analysis of the confinement and stability issues of a Crystal Beam. A method is described to determine the equilibrium configuration of a beam of highly charged particles. It is required that the beam has a uniform distribution in the direction of motion, which is stable and does not need therefore confinement with external means. It is shown that this can be obtained only for a relatively longitudinally compact beam. Confinement in the plane transverse to the direction of motion is obtained with external means, which also provide stability. It is important that particles are distributed so that the resulting space charge forces are linear with respect to the particle transverse coordinates.

I. INTRODUCTION

A Crystal Beam [1-3] is an ensemble of charged particles, all identical to each other, with the same electric charge Qe and mass at rest Am, where e is the electron charge and m the proton mass at rest. The charge state Q and mass number A are integer. Particles are treated point-like, with no internal structure. Only interaction among each other is the electromagnetic interaction. It is assumed that there is an equilibrium configuration where particles occupy a rigid position with respect to each other while all together move in one direction. Particles are allowed to oscillate around their equilibrium positions as long as the amplitude of the oscillations is small.

Confinement and stability questions are best described with the rectangular and infinite Crystal Beam model which is introduced in section II. We consider next the case of a cylindrical beam, infinitely long in the longitudinal direction, but having finite transverse dimensions. Section III defines this beam, whereas section IV discuss the requirement for the longitudinal stability and section V and VI the confinement and stability respectively in the transverse plane. It is seen that, in order to keep the beam confined transversely, a magnet with a profile providing focussing simultaneously in both transverse directions is required. This could be the case of a Betatron magnet. The field profile is also required to maintain the beam stable against transverse oscillations. Since the external restoring force is linear with the particle transverse coordinates, it is also important that the equilibrium configuration places particles in such a way that the resulting space charge forces are also linear.

Section VII defines quantitatively the limits of the beam spreads in momenta as evidence of crystallization. Finally section VIII is an analysis of the effects introduced by the insertions of drifts. The resulting storage ring lattice is to show stability at the two extremes: when the space charge is ignored and for the final state of the Crystal Beam.

* Work performed under the auspices of U.S. DOE

II. THE RECTANGULAR MODEL

The number of particles is infinite. They all move with the same velocity in the same direction. Particles are uniformly distributed, extending to infinity in all three dimensions. They are equally spaced from each other, sitting at the knots of a rectangular grid with step size $\lambda_{\parallel}$ in the longitudinal direction and $\lambda_{\perp}$ in the plane transverse to it. The field experienced by a particle is the sum of all the fields generated by the other particles. Because of the symmetry arrangement, the field is the same for all particles and identically equal to zero. Thus there is no interaction between particles and they are perfectly screened from each other. This is an equilibrium configuration which obviously does not need to be confined with external forces.

To determine if this configuration is also stable, we add a longitudinal perturbation of motion [3] to any particle and calculate the resulting field. For a small perturbation, after linearizing the field expression, we obtain that the particle perturbed performs longitudinal oscillations with angular frequency given by

$$\Omega_{\parallel}^2 = \frac{Q^2 e^2 g_{\parallel}(w)}{\lambda_{\perp}^3 \gamma^2 mA} \tag{1}$$

where $w = \lambda_{\parallel} \gamma / \lambda_{\perp}$ and, with $j = (j_1, j_2, j_3)$,

$$g_{\parallel}(w) = \Sigma_{j \neq 0} \frac{2 w^2 j_3^2 - j_1^2 - j_2^2}{(j_1^2 + j_2^2 + w^2 j_3^2)^{5/2}} \tag{2}$$

A similar result is obtained when a transverse perturbation is added. The resulting angular frequency $\Omega_{\perp}$ has the same expression of Eq. (1) but the response function $g_{\parallel}(w)$ is replaced by $g_{\perp}(w) = - g_{\parallel}(w) / 2$. Stability requires that both $g_{\parallel}$ and $g_{\perp}$ are positive; this cannot be satisfied at the same time. We choose $g_{\parallel} > 0$, so that the beam is stable longitudinally, and let the beam to be unstable in the transverse plane. An external restoring force is then required to recover the beam stability also in the transverse plane. The condition $g_{\parallel} > 0$ is satisfied for $w < 1$, that is for a longitudinally compact beam.

III. THE CYLINDRICAL MODEL

We continue assuming an infinitely long beam but with a more realistic finite cross-section. Particles are still uniformly distributed in the longitudinal direction, where they are separated again by the period $\lambda_{\parallel}$ so that no confinement is required and the motion is stable in that direction. We assume there is an equilibrium configuration where the beam is made of a number n_s of *shells* of elliptic cross-sections of the same aspect ratio. Each *shell* is made of n_h particles equally spaced by the same angle $\theta = 2\pi / n_h$. The innermost shell is a *string* where the particles are aligned, equally spaced, on the beam axis.

It can be seen that the longitudinal component of the field vanishes identically, whereas the transverse component is zero on the beam axis and increases in amplitude toward the edge of the beam. Its magnitude and direction is independent on the longitudinal position along the beam. In the limit of a large number N of particles distributed over a length L, and for a round beam of radius b, the transverse field is actually linear with the distance r from the beam axis [4], that is

$$E_\perp = 2QeNr / Lb^2 = kr \tag{3}$$

as long as the i-th shell is located at the radius $b_i = b\sqrt{i/n_s}$. In particular the *string* has vanishing radius, that is $b_0 = 0$.

Clearly this configuration needs to be confined with transverse external restoring forces, also linear.

IV. LONGITUDINAL STABILITY

As for the previous model, we determine the longitudinal stability by adding a small longitudinal perturbation to the motion of any particle. After linearizing the field expression, we derive that the particle performs longitudinal oscillations with angular frequency $\Omega_\parallel$ given by

$$\Omega_\parallel^2 = \frac{Q^2 e^2 g_\parallel(w)\, n_s^{3/2}}{b^3 \gamma^2 mA} \tag{4}$$

where now $w^2 = \gamma^2 \lambda_\parallel^2 n_s / b^2$ and ($i = 0, 1, ..., n_s$)

$$g_\parallel(w) = \Sigma_{j\neq i} \frac{2w^2 j_3^2 - i - j_1 + 2\sqrt{ij_1}\cos(\theta j_2)}{[i + j_1 - 2\sqrt{ij_1}\cos(\theta j_2) + w^2 j_3^2]^{5/2}} \tag{5}$$

In order for the beam to be stable against longitudinal perturbations it is required that $g_\parallel(w) > 0$, that is the compactness parameter w is to be less than a limiting value which depends on the number n_s of *shells* and on the separation angle θ. For instance [3], for a single *shell* $w < 1.9\,\theta$ and for ten *shells* $w < 6.0\,\theta$.

V. TRANSVERSE CONFINEMENT

Let us consider the motion of the i-th particle in the transverse plane by adding a radial perturbation [3]. We shall still assume a round beam, for simplicity. Its position can be written as $\boldsymbol{r}_i = \boldsymbol{r}_{0i} + \boldsymbol{u}$. The equation of motion can be written as follows

$$\begin{aligned} mA\,\ddot{\boldsymbol{r}}_i &= Qe\,\boldsymbol{E}_i(\boldsymbol{r}_i) \\ &= Qe\,\boldsymbol{E}_i(\boldsymbol{r}_{0i}) + Qe\, d\boldsymbol{E}_i(\boldsymbol{r}_i)/d\boldsymbol{r}_i \Big|_{\boldsymbol{r}_{0i}} \boldsymbol{u} + \ldots \qquad (6) \\ &= mA\,\ddot{\boldsymbol{r}}_{0i} + mA\,\ddot{\boldsymbol{u}} \end{aligned}$$

where $\boldsymbol{E}_i$ is the field acting on the particle. This can be broken down into two equations

$$mA\,\ddot{\boldsymbol{r}}_{0i} = Qe\,\boldsymbol{E}_i(\boldsymbol{r}_{0i}) \tag{7}$$

and

$$mA\,\ddot{\boldsymbol{u}} = Qe\, d\boldsymbol{E}_i(\boldsymbol{r}_i)/d\boldsymbol{r}_i \Big|_{\boldsymbol{r}_{0i}} \boldsymbol{u} \tag{8}$$

The field is made of two contributions: the *internal*, due to the beam proper, and the *equivalent external* due to the restoring forces. In particular,

$$\boldsymbol{E}_i = (k - k_{ext})\, \boldsymbol{r}_{0i} \tag{9}$$

where k is given by Eq. (3) and k_{ext} corresponds to the restoring forces.

The following *confinement condition* is to be satisfied

$$k_{ext} = k \tag{10}$$

This condition is fulfilled by having the beam circulating in a Betatron which provides focusing in both transverse directions at the same time. The magnetic field profile is measured by the field index n, a positive quantity, less than unit. For a round beam, a suitable choice is $n = 0.5$ in which case

$$k_{ext} = \beta B_0 / 2\rho \tag{11}$$

where B_0 is the bending field and ρ the bending radius.

The confinement condition can be expressed in terms of the magnet and beam parameters as follows

$$A\beta^2\gamma^2\pi b^2 = 2Q^2 r_0 N\rho \tag{12}$$

VI. TRANSVERSE STABILITY

The stability of motion in the transverse plane is investigated by solving Eq. (8) where

$$\frac{d\boldsymbol{E}_i(\boldsymbol{r}_i)}{d\boldsymbol{r}_i}\Bigg|_{\boldsymbol{r}_{0i}} = -k_{ext} - Qe\, g_\perp(w)\, n_s^{3/2} / b^3 \tag{13}$$

The first term at the right-hand side is the contribution from the external restoring forces whereas in the second term, proper of the beam, $g_\perp(w) = -g_\parallel(w)/2$. We recover thus the result we have already obtained for the rectangular Crystal Beam model. Therefore the same considerations made before will also apply here.

The following *stability condition* is to be satisfied

$$k_{ext} > Qe\, g_\parallel(w)\, n_s^{3/2} / 2b^2 \tag{14}$$

which can also be written as

$$w\, g_\parallel(w) < 4n_h \tag{15}$$

Particles will then perform transverse oscillations with an angular oscillation frequency $\Omega_\perp$ given by

$$\Omega_\perp^2 = \frac{2Q^2 e^2 (N/L)}{mA\gamma^3 b^2}\, \varepsilon(w) \tag{16}$$

where

$$\varepsilon(w) = 1 - w\, g_\parallel(w) / 4n_h \tag{17}$$

which ranges between 0 and 1 for the motion to be stable in the transverse plane.

VII. BEAM CRYSTALLIZATION

With a perturbation added, the beam will perform stable longitudinal oscillations provided that the amplitude $a_\parallel$ of the oscillations is small enough. The beam momentum spread is then measured by the maximum velocity encountered during the oscillations, that is

$$\Delta p_{\parallel} = mA\,\gamma\, a_{\parallel}\Omega_{\parallel} \qquad (18)$$

where $\Omega_{\parallel}$ is given by Eq. (4). Since the *condition for crystallization* can be taken as

$$a_{\parallel} < \lambda_{\parallel} \qquad (19)$$

the same condition can be expressed as follows

$$\Delta p_{\parallel} / p < \lambda_{\parallel}\,\Omega_{\parallel} / \beta c \qquad (20)$$

A similar condition ought also to be satisfied for the transverse momentum spread, that is

$$\Delta p_{\perp} < mA\,\gamma\,\theta\, b\,\Omega_{\perp} \qquad (21)$$

which is obtained by requiring that transverse oscillations have an amplitude $a_{\perp} < \theta b$, and where $\Omega_{\perp}$ is given by Eq.(16).

VIII. INSERTION OF DRIFT SPACES

Circulation of the beam in a betatron magnet will introduce the effect of *curvature* that we shall not investigate here. Drifts are required for beam manipulation like injection, abort and cooling. The betatron magnet will thus be broken in M identical periods each of length $\ell_B = 2\pi\rho / M$ and separated by drifts of length ℓ_D.The insertion of drifts will disrupt the *equilibrium configuration* and it will not be possible to maintain the beam cross-section constant. The motion remains periodic with the period length equal to $2\pi\rho / M + \ell_D$.

Let z_j denote either the horizontal or the vertical coordinate of the j-th particle. It is more convenient to replace the time as the independent variable with the curvilinear length s travelled along the reference closed orbit. The equation of motion can now be written as

$$z_j'' + [Qe(k_{ext} - k) / mA\,\gamma\,\beta^2 c^2]\, z_j = 0 \qquad (22)$$

where $k_{ext} = 0$ in the drifts and takes a constant value in the sector magnets. For convenience let us write

$$K = Qe\, k_{ext} / mA\,\gamma\,\beta^2 c^2 \qquad (23)$$

The term k which is proper of the beam itself, includes a dependence on the beam size b and therefore will also vary periodically. It is convenient to write the dependence with the beam size explicitly

$$Qe\, k / mA\,\gamma\,\beta^2 c^2 = \frac{2Q^2 e^2\,(N/L)}{mA\,\gamma^3\, b^2 \beta^2 c^2} = h / b^2 \qquad (24)$$

The solution of the equation above of course has to be periodic with the period given by $\ell_B + \ell_D$. In particular, the equation of motion can be written for the particle at the edge of the beam, thus deriving the equation of the beam envelope

$$b'' + Kb - h/b = 0. \qquad (25)$$

We shall assume that the drifts are not too long so that the beam dimension remains almost constant with a little periodic modulation added. Let $b = b_0(1 + \Delta)$ where b_0 is the beam size in absence of drifts. We shall expand and retain only terms linear in Δ. In the drift regions $K = 0$ and the envelope equation reduces to

$$\Delta'' + K_D\,\Delta = K_D \qquad (26)$$

with $K_D = h / b_0^2$. The equation for the sector magnet, letting $K_B = K + K_D$, is simply

$$\Delta'' + K_B\,\Delta = 0 \qquad (27)$$

The periodic solution of these equations, which are linear, can be searched with the conventional 3 x 3 matrix notation [3].

To determine the stability on the transverse plane, as usual, we add a small perturbation u to any particle along one of the two transverse directions. After linearization, the equation of motion is

$$u'' + q\,u = 0 \qquad (28)$$

where, in the sector magnets,

$$q = \Omega_{\perp}^2 / \beta^2 c^2 = q_B \qquad (29)$$

and, in the drifts,

$$q = -q_B\, w\, g_{\parallel}(w) / 4\, n_h\, \varepsilon(w) = -q_D \qquad (30)$$

As usual, we can search the solution with a matrix notation [3]. Only 2 x 2 matrices are required. The motion is stable if the trace of the transfer matrix corresponding to a period has an absolute value not exceeding 2, that is

$$|2\cos\xi_B \cosh\xi_D + (\sqrt{q_D/q_B} - \sqrt{q_B/q_D})\sin\xi_B \sinh\xi_D| < 2 \qquad (31)$$

where $\xi_B = \ell_B\sqrt{q_B}$ and $\xi_D = \ell_D\sqrt{q_D}$. A similar condition is to be satisfied also for the stability of motion in the storage ring in the limit of zero space-charge forces, that is

$$|\,2\cos(\ell_B/\rho) - (\ell_D/\rho)\sin(\ell_B/\rho)\,| < 2 \qquad (32)$$

IX. ACKNOWLEDGMENTS

The author wishes to thank the scientific staff of the Laboratori Nazionali di Legnaro (Padua, Italy) for many useful and informative discussions. This work is a contribution to the CRYSTAL collaboration.

X. REFERENCES

[1] J.P. Schiffer and A. Rahman, "Feasibility of a Crystalline Condensed State in Cooled Ion Beams of a Storage Ring", PHY-5121-ME-88, Feb. 1988, Argonne National Laboratory.

[2] Proceedings of the Workshop on Crystalline Ion Beams, Werheim, Germany, Oct. 1988. GSI-89-10 Report.

[3] A.G. Ruggiero, "Confinement and Stability of a Crystal Beam", BNL Formal Report, in preparation, May 1993. AGS Department, Brookhaven National Laboratory.

[4] A.G. Ruggiero, "Review of Space Charge Calculations", Booster Technical Note No. 104, January 1988. AD Department, Brookhaven National Laboratory.