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A High-Order Moment Simulation Model*

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1. Introduction

The physical length of accelerator systems presents computational difficulties for three-dimensional discrete-particle simulations of the dynamics and emittance growth in beam transport. An alternative to the discrete-particle approach is a high-order moment equation model, which is an extension of envelope equations formulated first by Kapchinskij and Vladimirskij[1] and later adapted by Lee, Close and Smith[2].

Chernin[3] has derived a beam envelope equation from the equations of motion for a mono-energetic particle beam in a magnetic field that is linear in the transverse coordinates. In this work we are interested in beams with a spread in energy, and transport systems with a non-linear transverse dependence of the magnetic field. For this type of system the spatial coordinate along the beam motion cannot be used as the time variable; faster particles will overtake the slower particles as time evolves. We present relativistically covariant moment equations for modeling beam transport, based on the work of Newcomb[4] and Amendt and Weitzner[5]. The beam is described by a set of partial differential moment equations, instead of a set of ordinary differential envelope equations. Our formalism is based on transverse averages of the moment equations obtained from the relativistic Vlasov equation. The spatial coordinate along the beam motion and time are the only independent variables. The moment equations are closed by setting higher order correlation functions to zero. A similar formulation of moment equations by Channel and co-workers[6] used spatial averaging in all three coordinates to model a bunched beam with time as the only independent variable. For bunch lengths that are large compared to the betatron wavelength, it is impractical to carry sufficiently high longitudinal moments to model the oscillations within the bunch.

2. Formuation

We denote the time t and local Cartesian space coordinate (x^1, x^2, x^3), replacing the usual coordinate (x, y, z), where x^3 is measured along the beam motion direction and x^1 and x^2 are the transverse directions. We define $x^4 = ct$, where c is the speed of light, so that space-time is parametrized by $x^\mu, \mu = 1,2,3,4$. We use a summation convention, and assume that Latin subscripts and superscripts, i, j, k, l, are summed from one to three, while Greek subscripts and superscripts are summed from one to four. The space-time metric $(ds)^2 = dx^i dx^i - c^2(dt)^2$ becomes $(ds)^2 = dx^\mu dx^\nu g_{\mu\nu}$, where the non-zero elements of the metric tensor $g_{\mu\nu}$ are $g_{ij} = \delta_{ij}$ and $g_{44} = -1$. The metric tensors $g_{\mu\nu}$ and $g^{\mu\nu}$, which is defined so that $g_{\mu\nu}g^{\nu\lambda} = \delta_\mu^\lambda$, are used to to raise and lower indices covariantly. The usual three velocity v^i is extended to a relativistic covariant four-velocity u^μ by the definitions $\gamma^{-2} = 1 - v^i v^i/c^2$ and $u^i = \gamma v^i, u^4 = \gamma c$ so that $u^\mu u_\mu = -c^2$.

The electromagnetic field tensor $F_{\mu\nu}$ is antisymmetric and is given by

$$E_i = cF_{i4} = -cF_{4i},$$
$$B_1 = F_{23} = -F_{32}, B_2 = F_{31} = -F_{13}, B_3 = F_{12} = -F_{21},$$

while the Lorentz force on a particle of charge q is $q(\vec{E} + \vec{v} \times \vec{B})_i = qF^{i\mu}u_\mu/\gamma$. The general form of the external magnetic field of interest can be expressed as

$$B_i = B_{i0} + B_{i1}x^1 + B_{i2}x^2 + B_{i11}x^1x^1 + B_{i12}x^1x^2 + B_{i22}x^2x^2,$$

where all the coefficients $B_{10}, B_{20}, B_{30}, B_{11}, ..., B_{222}$ are functions of x^3, with B_{i0} the dipole, B_{ij} the quadrupole, and B_{ijk} the sextupole components. The beam distribution function, $f(x^\mu, u^i)$, satisfies the relativistic Vlasov equation, which we express in covariant form as:

$$\left(u^\mu \frac{\partial}{\partial x^\mu} + \frac{q}{m}F^{i\mu}u_\mu \frac{\partial}{\partial u^i}\right) f = 0 \quad . \tag{1}$$

where m is the particle mass. The volume element $d\omega = du^1 du^2 du^3/\gamma$ in the four-momentum space is invariant under a Lorentz transformation. Since the transverse coordinates, x^1 and x^2, are invariant under a Lorentz transformation, we define an invariant phase space volume element under a Lorentz transformation to be $d\Omega = dx^1 dx^2 du^1 du^2 du^3/\gamma$, and a phase space average $\langle X \rangle = h^{-1} \int X f d\Omega$, with $h = \int f d\Omega$. We also define a second order correlation function:

$$[u^\mu u^\nu] = h^{-1} \int f(u^\mu - \langle u^\mu \rangle)(u^\nu - \langle u^\nu \rangle) d\Omega, \tag{2}$$

and similar definitions for the third order correlation functions.

The lowest moment of the Vlasov equation gives

$$\frac{\partial}{\partial x^3} h\langle u^3 \rangle + \frac{\partial}{\partial x^4} h\langle u^4 \rangle = 0. \tag{3}$$

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With Eq. 2 multiplied by u^ν and $u^\nu u^\lambda$ then integrated over $d\Omega$, we have:

$$\frac{\partial}{\partial x^3}h\langle u^3u^\nu\rangle + \frac{\partial}{\partial x^4}h\langle u^4u^\nu\rangle = \frac{q}{m}h\langle F^{\nu\mu}u_\mu\rangle, \qquad (4)$$

$$\frac{\partial}{\partial x^3}h\langle u^3u^\nu u^\lambda\rangle + \frac{\partial}{\partial x^4}h\langle u^4u^\nu u^\lambda\rangle = \frac{q}{m}h(\langle F^{\nu\mu}u_\mu u^\lambda\rangle + \langle F^{\lambda\mu}u_\mu u^\nu\rangle). \qquad (5)$$

There are four independent equations represented in Eqs. 4 and ten in Eqs. 5. Equations 3 to 5 are basically the same as the fluid equations of Newcomb[4] and Amendt and Weitzner[5], with the additional averaging over the transverse coordinates. Since $F^{\mu\nu}$ depends on the transverse coordinates, Eqs. 4 and 5 cannot be closed without introducing the spatial moment equations:

$$\frac{\partial}{\partial x^3}h\langle u^3x^i\rangle + \frac{\partial}{\partial x^4}h\langle u^4x^i\rangle = h\langle u^i\rangle, \qquad (6)$$

$$\frac{\partial}{\partial x^3}h\langle u^3u^\nu x^i\rangle + \frac{\partial}{\partial x^4}h\langle u^4u^\nu x^i\rangle = h\langle u^\nu u^i\rangle + \frac{q}{m}h\langle F^{\nu\mu}u_\mu x^i\rangle, \qquad (7)$$

$$\frac{\partial}{\partial x^3}h\langle u^3x^ix^j\rangle + \frac{\partial}{\partial x^4}h\langle u^4x^ix^j\rangle = h\langle x^ju^i\rangle + h\langle x^iu^j\rangle, \quad (8)$$

for $i, j = 1, 2$ only.

Physical meanings can be attached to these moments. The first order moments $\langle x^1\rangle$ and $\langle x^2\rangle$ denote the centroid position, and $\langle u^1\rangle, \langle u^2\rangle, \langle u^3\rangle$ and $\langle u^4\rangle$ are associated with the beam current and density respectively. The second order spatial correlations $[x^ix^j]$ with $i, j = 1, 2$ define the transverse beam envelope ellipse. The second order momentum correlations $[u^\nu u^\mu]$ are the thermal momentum/energy spread. The second order cross correlations $[x^iu^\nu]$ are the current and density dipole moments.

To allow easier numerical solution, consider a new general variable y^λ such that $y^1 = 1$, $y^2 = x^1$, $y^3 = x^2$, $y^4 = u^1$, $y^5 = u^2$, $y^6 = u^3$, and $y^7 = u^4$. The twenty-eight equations represented in Eq. 3 thru 8 can be obtained by multiplying Eq. 1 by $y^\lambda y^\nu$ and integrating over $d\Omega$.

$$\frac{\partial}{\partial x^3}h\langle u^3y^\lambda y^\nu\rangle + \frac{\partial}{\partial x^4}h\langle u^4y^\lambda y^\nu\rangle = \Gamma(\lambda,\nu) + \Gamma(\nu,\lambda) \quad (9)$$

where

$$\Gamma(\lambda,\nu) = \begin{cases} 0, & \lambda = 1; \\ \int y^\nu u^{\lambda-1} f d\Omega, & \lambda = 2 \text{ or } 3; \\ \frac{q}{m}\int y^\nu F^{\lambda-3,\mu}u_\mu f d\Omega, & \lambda = 4 \text{ thru } 7. \end{cases} \qquad (10)$$

To close the second order system of equations we assume that correlations above second order are zero. This still allows third order moments to be nonzero. The closing condition creates several equivalent families of independent variables: $\{h\langle y^\lambda y^\nu u^3\rangle\}$ and $\{h\langle y^\lambda y^\nu u^4\rangle\}$ where $\lambda = 1$ to 7 and $\nu = \lambda$ to 7, $\{h, \langle y^\lambda y^\nu\rangle\}$ where $\lambda = 2$ to 7 and $\nu = \lambda$ to 7, or $\{h, \langle y^\lambda\rangle, [y^\lambda y^\nu]\}$ where $\lambda = 2$ to 7 and $\nu = \lambda + 1$ to 7. Note that each set has twenty-eight elements. The natural set of independent variables to advance in time is $\{h\langle y^\lambda y^\nu u^4\rangle\}$. At each cell in x^3 these 28 variables describe the moments of f when correlations above second order are zero. To solve this system of equations we need to calculate $h\langle u^3y^\lambda y^\nu\rangle$ and $\Gamma(\lambda,\nu)$ after each time advance. We need to relate $\{h\langle y^\lambda y^\nu u^4\rangle\}$ to the 84 non-zero third order moments, $\{h, \langle y^\lambda y^\nu y^\alpha\rangle\}$. We define $[\![\lambda,\nu]\!] \equiv h\langle y^\lambda y^\nu u^4\rangle$, and get from the third order correlation functions,

$$[\![\lambda,\nu]\!] = h[y^\lambda y^\nu u^4] + [\![1,1]\!]\langle y^\lambda y^\nu\rangle + [\![1,\nu]\!]\langle y^\lambda\rangle + [\![1,\lambda]\!]\langle y^\nu\rangle - 2[\![1,1]\!]\langle y^\lambda\rangle\langle y^\nu\rangle. \qquad (11)$$

With this a mapping from $[\![\lambda,\nu]\!]$ to $\{h, \langle y^\lambda y^\nu y^\alpha\rangle\}$ is derived. To close the n'th order system of equations we assume that correlations above order n are zero. Each order has a different closing condition, thus each order has a different mapping from $\{h\langle y^{\lambda_1}y^{\lambda_2}\cdots y^{\lambda_n}\rangle\}$ to $\{h, \langle y^{\lambda_1}y^{\lambda_2}\cdots y^{\lambda_{n+1}}\rangle\}$.

For second and third order systems, the systems have 28 and 84 equations respectively. Currently our computer model allows a fourth order system, which has 210 equations. The general expression for the total number of equations in an n-th order system can be expressed as:

$$1 + \sum_{m=1}^{n}\sum_{i=1}^{min(6,m)} C_i^6 C_{i-1}^{m-1}.$$

This gives 462 for the fifth order, and 924 for the sixth order systems. The advantage of Eqs.(9)-(11) is that they can be easily manipulated symbolically.

3. Space Charge Model

A space charge model has been implemented to include the image charges of the metallic boundary and the longitudinal component of the space charge fields. The model assumes that the charge density, ρ, can be approximated by a two-dimensional distribution of charged rods inside a cylindrical metallic pipe;

$$\rho = \sum_{i=1}^{N} q_i(x_3,t)g(\vec{x} - \vec{x_i}), \qquad (12)$$

where g is the spatial distribution function for the finite size charged rods whose location are independent of x_3. Note that g depends on x_1, and x_2, while q_i depends only on x_3. The charges on the rods, q_i, are chosen to be consistent with the spatial moments. In the second order moment system there are six spatial moments, and therefore we have $N = 6$ and a matrix equation to relate the coefficients q_i with the spatial moments: $hX = MQ$, where M is a 6×6 matrix whose elements are of the form $\int x_k^m x_l^n g(\vec{x} - \vec{x_i})dx_1dx_2$, with

$k, l = 1, 2$ and $m + n \leq 2$, X and Q are column matrices such that $X^T = (1, \langle x_1\rangle, \langle x_2\rangle, \langle x_1x_1\rangle, \langle x_2x_2\rangle, \langle x_1x_2\rangle)$ and $Q^T = (q_1, ..., q_6)$. Equation (15) can be easily inverted to express q_i in terms of the spatial moments.

The image charges of these rods are easy to determine. For a charge rod located at $\vec{x_i}$ with a charge q_i the image is located at $\vec{x_i'}$ with charge $-q_i$, where $\vec{x_i'}$ is along the same direction outside the metallic cylinder with a magnitude $a^2/|\vec{x_i}|$, and a is the radius of the cylinder. The electric field due to the charged rods inside the cylinder in the beam frame can be written as

$$\vec{E}(\vec{x}) = \sum_{i=1}^{N} q_i \frac{\vec{x} - \vec{x_i}}{|\vec{x} - \vec{x_i}|} - \sum_{i=1}^{N} q_i \frac{\vec{x} - \vec{x_i'}}{|\vec{x} - \vec{x_i'}|}. \qquad (13)$$

This result can be transformed back to the laboratory frame. With this simplified model, the self-field contribution to moments involving $F^{\nu\mu}$ can be readily expressed in terms of other moments retained in the system and a close set of equations can be achieved.

5. Results

The High-Order Moment (HOM) model agrees with SAIC's "ABBY"[3] code, a linear envelope model for the steady state evolution of a monoenergetic beam. To test the effect of energy spread in the HOM model, we injected two monoenergetic circular beams at $z = 0$ with γ's of 3 and 3.1, respectively, into a mismatched constant guide field. Each beam will independently exhibit betatron oscillations in the beams radius as a function of the distance of propagation, z. The space charge model has been turned off to eliminate any interaction between the two beams; the addition of the results from an envelope equation will be the exact solution. Figure 1a shows the expected beat pattern in the $\langle x^1x^1\rangle$ moment caused by the slight difference in frequencies of the betatron oscillations. The HOM model, when retaining up to third order correlations, does not capture this energy mixing. The average of the two betatron frequencies developes in time as shown in figures 1b and 1c. To capture the effects of this energy spread fourth order correlations must be retained (fig. 1d).

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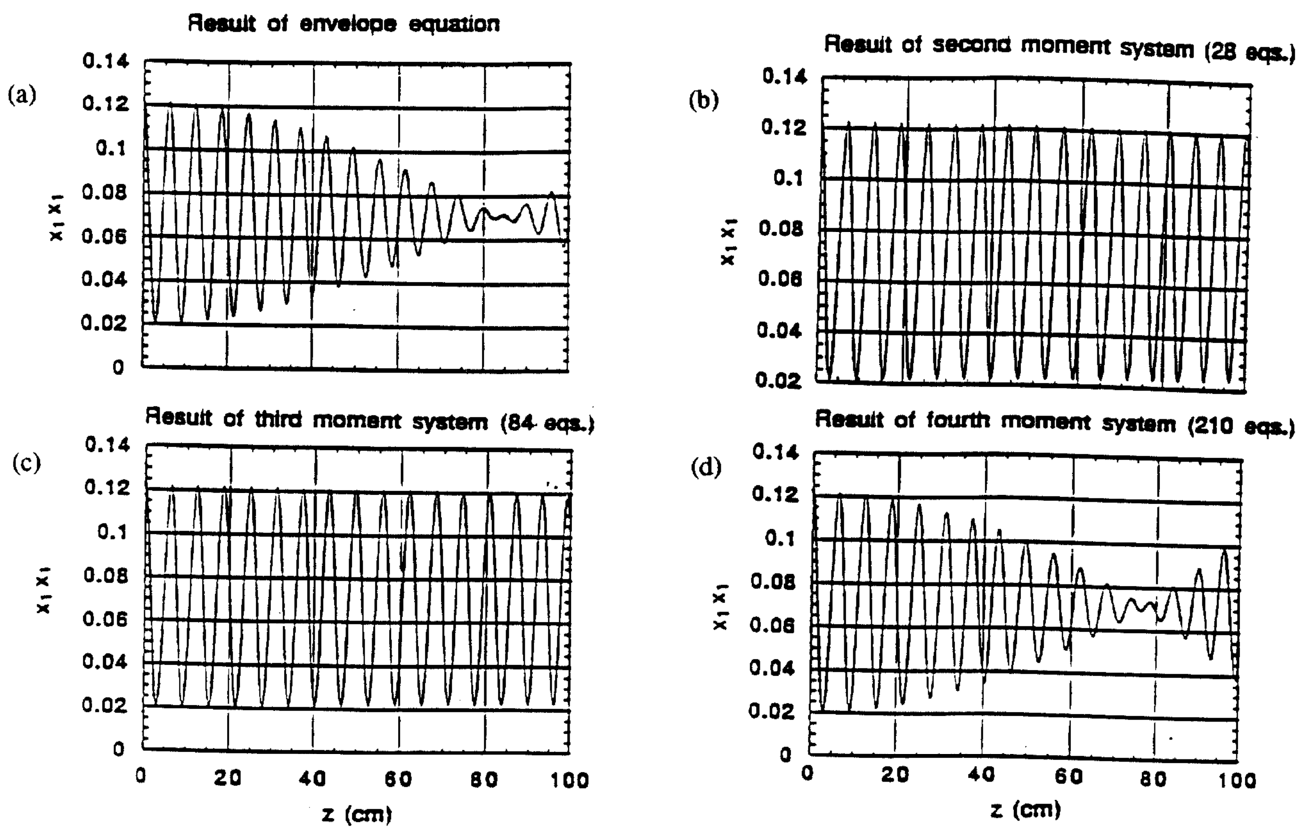

Fig. 1: Envelope solutions for a beam with energy spread.