# Analytical Formulae for the Coupling Coefficient $\beta$ between a Waveguide and a Travelling Wave Structure

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*Abstract*

Analytical formulae for the coupling coefficient $\beta$ of a waveguide-travelling wave structure coupling system have been established and compared with experimental results. The dimension of the coupling aperture on the coupler cavity of a travelling wave structure can be determined easily. The relation between the coupling coefficient $\beta$ and the other parameters, such as aperture dimension, wavelength, group velocity, wall thickness and coupler dimension, is explicitly revealed. These formulae include that for the case of a waveguide-single cavity coupling system which has been established in ref. [1].

## I INTRODUCTION

During the construction of a linear accelerator structure the determination of the dimension of the coupling aperture on the coupler cavity wall is always made by experiments. Recently the three dimensional program MAFIA has been used to design the coupler of a travelling wave structure [2]. Based on Bethe theory [3], however, in this paper analytical formulae which can be used to determine the coupling aperture's dimension are established and compared with experimental result. These formulae apply also to the case of a waveguide coupled to a single standing wave cavity.

## II BASIC THEORY

In a linear accelerator the electromagnetic energy is fed from a klystron through a waveguide to the accelerating structure. The coupling is effected by an aperture located on the common wall between the waveguide and one cavity, so-called coupler cavity, of the structure. If the linear dimension of this aperture is small compared with the wavelength, Bethe theory states that the aperture is equivalent to a combination of radiating electric and magnetic dipoles, whose dipole moments are proportional to the normal electric field and the tangential magnetic field of the incident wave, respectively [3]. If the aperture is small, static field solutions for the equivalent dipole moments of elliptic- and circular-shaped apertures can be found by placing these kind of apertures in static electric and magnetic fields [4].

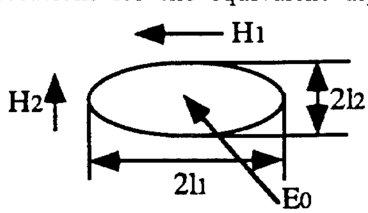



Figure 1: Coupling aperture

$$P = -\frac{\pi l_1^3(1-e_0^2)}{3E(e_0)}\epsilon E_0 \qquad (1)$$

$$M_1 = \frac{\pi l_1^3 e_0^2}{3(K(e_0)-E(e_0))}H_1 \qquad (2)$$

$$M_2 = \frac{\pi l_1^3 e_0^2(1-e_0^2)}{3(E(e_0)-(1-e_0^2)K(e_0))}H_2 \qquad (3)$$

$$e_0 = (1-l_2^2/l_1^2)^{1/2} \qquad (4)$$

where $\epsilon$ is the permittivity of the vacuum. $P$ and $M_1$ ($M_2$) are the electric and magnetic dipole moments, respectively. $E_0$ is the electric field perpendicular to the surface of the ellipse. $H_1$ and $H_2$ are the magnetic fields parallel to the major and minor axes of this ellipse. $l_1$ and $l_2$ are the lengths of the half-major and half-minor axes, respectively (see Fig. 1). $K(e_0)$ and $E(e_0)$ are complete elliptic integrals of the first and second kinds [5].

$$K(e_0) = \frac{\pi}{2}(1+(\frac{1}{2})^2e_0^2+(\frac{1\cdot 3}{2\cdot 4})^2e_0^4+(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6})^2e_0^6 + (\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8})^2e_0^8+\cdots) \qquad (5)$$

$$E(e_0) = \frac{\pi}{2}(1-(\frac{1}{2})^2e_0^2-(\frac{1\cdot 3}{2\cdot 4})^2\frac{e_0^4}{3}-(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6})^2\frac{e_0^6}{5} - (\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8})^2\frac{e_0^8}{7}-\cdots) \qquad (6)$$

For values of $e_0$ approching unity,

$$K(e_0) = ln(4\frac{l_1}{l_2}) \qquad (7)$$

$$E(e_0) = 1 \qquad (8)$$

Obviously, for $e_0$ equal to zero, the aperture becomes circular with

$$P = -\frac{2}{3}l^3\epsilon E_0 \qquad (9)$$

$$M_1 = M_2 = \frac{4}{3}l^3 H_{1,2} \qquad (10)$$

where $l = l_1 = l_2$. It should be mentioned that the apertures discussed above have no volumes, only elliptic surfaces, and that all the quantities used in this paper are in MKS units.

## III CALCULATION OF THE COUPLING COEFFICIENT $\beta$

A waveguide-travelling wave structure coupling system can be equivalent to the case of a waveguide-coupler cavity coupling system where the coupler cavity is coupled to the rest of the travelling wave structure. The coupling coefficient $\beta$ is defined as

$$\beta = \frac{P_e}{P_0 + P_t} \tag{11}$$

where $P_e$ is the power lost in the matched load of the waveguide, $P_0$ is the power dissipated inside the coupler cavity and $P_t$ is the power flow from the coupler cavity to the adjacent travelling wave structure. $P_t$ can be also expressed as

$$P_t = \frac{U}{L} v_g \tag{12}$$

where $U$ and $L$ are the stored energy and the length of the coupler cavity, respectively, and $v_g$ is the group velocity of the travelling wave structure. In terms of equivalent circuit, $\beta$ can also be expressed as

$$\beta = \frac{Y_c}{n^2 G_0} \tag{13}$$

where $Y_c$ is the admittance of the waveguide, $G_0$ is the equivalent resonant admittance of the cavity and $n$ is the transform ratio of an ideal transformer. Throughout this paper we consider only the lowest propagation mode $H_{10}$ in the waveguide. In terms of Power-Voltage definition,

$$Y_c = \frac{a(1-(\frac{\lambda}{2a})^2)^{1/2}}{2Z_0 b} \tag{14}$$

$$G_0 = \frac{2(P_0 + P_t)}{V_0^2} \tag{15}$$

where $a$ and $b$ are the width and height of the waveguide, respectively, $\lambda$ is the wavelength in free space, $Z_0 = 120\pi$ (Ohm) and $V_0$ is the equivalent voltage across the gap of the cavity. From the definition of an ideal transformer

$$\frac{1}{n} = \frac{V_1}{V_0} \tag{16}$$

where $V_1$ is the equivalent voltage induced in the waveguide by means of the coupling aperture.

If the coupling aperture is located where there is only magnetic field as shown in Fig. 2, the coupling of energy is

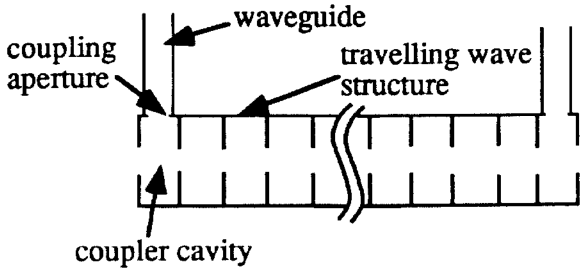



Figure 2: Magnetic coupling

performed by $M_1$ or $M_2$. The $H_1$ and $H_2$ in eq. 2 and eq. 3 are the magnetic fields on the wall of the cavity when the aperture is replaced by a metal surface, that is to say $H_1$ or $H_2$ can be calculated by a 2-D program such as Superfish if the cavity is cylindrically symmetric. According to the theory stated in ref. [4], if compared with the size of the aperture the cavity wall can be considered as plane rather than bent, the amplitude of the $H_{10}$ mode magnetic field $H_0$ induced in the waveguide by the equivalent magnetic dipole moment of the aperture is expressed as

$$H_0 = j\omega\mu_0 h_{10}^2 M_{1,2} \tag{17}$$

where

$$h_{10} = \Gamma_{10}(\frac{2}{jabk_0 Z_0 \Gamma_{10}})^{1/2} sin(\frac{\pi x}{a}) \tag{18}$$

$$k_0 = \frac{2\pi}{\lambda} \tag{19}$$

$$\Gamma_{10} = jk_0(1-(\frac{\lambda}{2a})^2)^{1/2} = jk_{10} \tag{20}$$

The equivalent voltage $V_1$ induced in the waveguide could be expressed as

$$V_1 = E_0 b = \frac{Z_0 k_0}{k_{10}} H_0 b \tag{21}$$

where $E_0$ is the electric field of $H_{10}$ mode. If the coupling aperture is located at the center of the waveguide cross section, that is to say $x = a/2$ in eq. 18, using all the equations in this section eq. 13 can be expressed as

$$\beta_1 = \frac{\pi^2 Z_0 k_0 k_{10} e_0^4 l_1^6}{9ab(K(e_0) - E(e_0))^2} \frac{H_1^2}{P_0 + P_t} \tag{22}$$

or

$$\beta_2 = \frac{\pi^2 Z_0 k_0 k_{10} e_0^4 (1-e_0^2)^2 l_1^6}{9ab(E(e_0) - (1-e_0^2)K(e_0))^2} \frac{H_2^2}{P_0 + P_t} \tag{23}$$

where $\beta_1$ and $\beta_2$ correspond to $M_1$ and $M_2$, respectively. It should be noticed that eq. 22 and eq. 23 do not include the effect of the wall thickness of the coupling aperture. Practically, however, it is necessary to take this effect into account. It is natural to imagine that the cavity is connected with the waveguide by a small section of cylindrical or rectangular waveguide with corresponding cutoff wavelength $\lambda_c$ depending on the shape of coupling aperture. If the thickness of the coupling aperture is $d$, the electromagnetic power after passing through this aperture will be reduced by a factor $e^{-2\alpha d}$ where

$$\alpha = k_0((\frac{\lambda}{\lambda_c})^2 - 1)^{1/2} \tag{24}$$

where $\lambda \geq \lambda_c$. By joining this factor to the right side of eq. 22 and eq. 23, the final analytical formulae of $\beta$ are established as

$$\beta_1 = \frac{\pi^2 Z_0 k_0 k_{10} e_0^4 l_1^6 e^{-2\alpha d}}{9ab(K(e_0) - E(e_0))^2} \frac{H_1^2}{P_0 + \frac{U}{L} v_g} \tag{25}$$

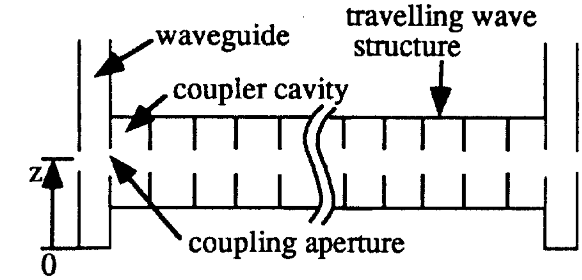



Figure 3: Electric coupling

or

$$\beta_2 = \frac{\pi^2 Z_0 k_0 k_{10} e_0^4 (1-e_0^2)^2 l_1^6 e^{-2\alpha d}}{9ab(E(e_0) - (1-e_0^2)K(e_0))^2} \frac{H_2^2}{P_0 + \frac{U}{L} v_g} \qquad (26)$$

where the $\alpha$ in eq. 25 and eq. 26 may not be same. For a cylindrical symmetric coupler cavity the cavity parameters $\frac{H_1^2}{P_0}$, $\frac{H_2^2}{P_0}$, $\frac{H_1^2}{U}$ and $\frac{H_2^2}{U}$, can be calculated by Superfish and for a cylindrically asymmetric cavity three-dimensional program such as MAFIA or PRIAM have to be used. Right now, once the group velocity and the aperture dimension are given one can directly calculate the coupling coefficient $\beta$.

If the coupling is effected by electric field as in the case shown in Fig. 3, the coupling coefficient $\beta$ is found to be

$$\beta = \frac{\pi^2 k_0^3 (1-e_0^2)^2 l_1^6 e^{-2\alpha d} sin^2(\lambda_g z)}{9ab Z_0 k_{10} E(e_0)^2} \frac{E_0^2}{P_0 + \frac{U}{L} v_g} \qquad (27)$$

where $E_0$ is the electric field on the surface of the aperture before it is opened, $\lambda_g$ is the wavelength in the waveguide ($H_{10}$ mode) and $z$ is the distance from the end of the waveguide to the center of the coupling aperture.

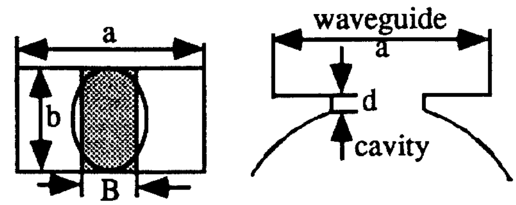
It is obvious that when the group velocity $v_g = 0$ the formulae derived above correspond to the case of waveguide-single cavity coupling system [1].

## IV COMPARISON WITH EXPERIMENTAL RESULT

In this section we will compare the analytical result with that of an experiment. We take the output coupler of LIL accelerating structure (which is used as injector for LEP at CERN) as an example. The coupler cavity parameters are listed in Table 1. The rectangular coupling aperture is equivalent to an elliptic one as shown in Fig. 4. The rectangular aperture can be regarded as a section of waveguide of length $d$ working at $H_{01}$ mode with cut-off wavelength $\lambda_c = 2B$, where the wall thickness $d$ is shown in Fig. 4. Fig. 5 shows the comparison between the experimental result and that analytically obtained from eq. 26.

Table 1: Parameters of LIL output coupler cavity

| U(J) | L(cm) | $c/v_g$ | $P_0$(W) | $H_2$(A/m) | d(cm) |
|---|---|---|---|---|---|
| 0.00107 | 3.4 | 147 | 1375 | 3420 | 0.27 |




Srect.=Bb=Sellip.=πl1l2, a=7.2 cm, b=3.4cm

Figure 4: Coupling aperture of LIL structure

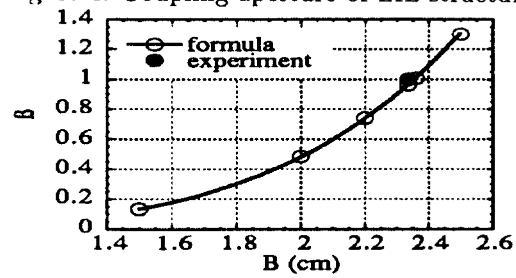



Figure 5: Comparison between analytical and experimental results

## V OTHER APPLICATIONS

Another important application of these analytical formulae is to estimate the loaded $Q$ of higher order modes in a damped structure (either a travelling wave structure or a single cavity) when a waveguide is connected to the structure to take out the power of higher order modes [6]

$$Q_L = \frac{Q_0}{1+\beta} \qquad (28)$$

where the dimension of the waveguide should be chosen as that the frequency of the accelerating mode is below the cut-off frequency of $H_{10}$ mode in the waveguide.

## VI ACKNOWLEDGEMENTS

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