Taylor Map and Calculation of Equilibrium Polarization Direction for Proton Storage Rings

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Abstract

The spin-orbital motion in proton storage rings is investigated with help of the methods of Lie operator and computer algebra. The one-turn transfer map is used to obtain the orbital and spin tunes, the invariant functions of the orbital motion and the equilibrium polarization direction.

I. Hamiltonian Form of Classical Spin-Orbit Motion Equations

The system of the ordinary differential equations is called a Hamiltonian system if it can be represented in the form

$$\frac{d\vec{z}}{d\tau} = \{\vec{z}, H\} \tag{1}$$

where $H = H(\vec{z}, \tau)$ is the Hamiltonian, the binary operation $\{\cdot,\cdot\}$ is the Poisson bracket with the usual properties.

The classical spin-orbit equations of the motion in storage rings or in circular accelerators have the form of a Hamiltonian system if we use the Poisson bracket:

$$\{F(\vec{z}), Q(\vec{z})\} = F_{\vec{q}} \cdot Q_{\vec{p}} - F_{\vec{p}} \cdot Q_{\vec{q}} + [F_{\vec{s}} \times Q_{\vec{s}}] \cdot \vec{S} \tag{2}$$

and the Hamiltonian:

$$H = H_{orbt}(\vec{x}, \tau) + \vec{W}(\vec{x}, \tau) \cdot \vec{S} \tag{3}$$

where $\vec{z} = (\vec{x}, \vec{S})$ and $\vec{x} = (\vec{q}, \vec{p})$ are canonical orbit variables, $\vec{S} = (S_1, S_2, S_3)$ is a classical spin vector.

For example, if τ means the time t, $\vec{q}$ and $\vec{p}$ are canonical orbital position and momentum variables in a fixed Cartesian coordinate system, then

$$H = e\Phi + c\sqrt{m_0^2c^2 + (\vec{p} - \frac{e}{c}\vec{A})^2} + \vec{W} \cdot \vec{S}$$

where

$$\vec{W} = -\frac{e}{m_0\gamma c}((1+\gamma G)\vec{B} - \frac{G\gamma^2}{(1+\gamma)c^2}(\vec{v}\vec{B})\vec{v} - (G\gamma + \frac{\gamma}{1+\gamma})[\vec{v} \times \frac{\vec{E}}{c}])$$

e, m_0 are the charge and the rest mass of a particle, c is the velocity of light, $\vec{E}$, $\vec{B}$ are electric and magnetic fields, $G = (g-2)/2$, g is the anomalous spin factor, γ is the Lorentz factor, $\vec{v}$ is the velocity of a particle, $\vec{A}$ and Φ are vector and scalar potentials [1].

The Poisson bracket (2) is degenerated. It has the nontrivial Casimir function $|\vec{S}|^2$. It means that we can decrease the dimension of the system (1) studding the equations on the sphere: $|\vec{S}| = \hbar/2$, where $\hbar$ is Planck's constant. For example, we can do this by introducing a pair of canonical spin variables (J, ψ), J is a projection on the selected axis, ψ is a polar angle in the transverse plane. In this case we obtain the Hamiltonian system with the classical Poisson bracket in eight-dimensional phase space (6 orbital variables and two canonical spin variables J, ψ). But unfortunately, in this case we lose so important property which we would like to keep: the linearity of the initial system in respect of spin variables.

II. One Turn Map and Its Simple Properties

If the Hamiltonian (3) is a periodic function τ, we can consider the one turn map which is defined by the action of the system (1) for one period:

$$\vec{z}_f = \vec{f}(\vec{z}_i) \tag{4}$$

Let $\vec{z} = \vec{0}$ be the fixed point of the map (4). Using the Taylor series expansion of the function $\vec{f}(\vec{z})$ in respect of spin variables we obtain:

$$\begin{cases} \vec{x}_f = \vec{F}(\vec{x}_i) + O(|\vec{S}_i|) \\ \vec{S}_f = A(\vec{x}_i) \cdot \vec{S}_i + O(|\vec{S}_i|^2) \end{cases} \tag{5}$$

where $\vec{F}(\vec{0}) = \vec{0}$ and $A(\vec{x})$ is a 3×3 matrix. The map (5) preserves the Poisson bracket (2). Using this property we find:

a) The map $\vec{x}_f = \vec{F}(\vec{x}_i)$ is symplectic:

$$\left(\frac{\partial \vec{F}}{\partial \vec{x}}\right)^{\perp} \cdot J \cdot \left(\frac{\partial \vec{F}}{\partial \vec{x}}\right) = J$$

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b) Every element of the matrix $A(\vec{x})$ is equal to its own cofactor. It means that $A(\vec{x})$ is an orthogonal matrix and $detA(\vec{x}) = 1$, i.e. $A(\vec{x}) \in SO(3)$.

III. Lie Exponents, Order Truncation, and Exponential Factorization

Using the Poisson bracket (2) we can define the Lie operator $: Q(\vec{z}) :$ and the exponential Lie operator $exp(: Q(\vec{z}) :)$ associated with the function $Q(\vec{z})$ according to the usual rules [2].

The length of the vector $\vec{S}$ is proportional to Planck's constant. Consequently, we will neglect terms of order $\hbar$ in the map (5). It is possible to show that for any given truncation order of the orbital variables m there is the Dragt-Finn's type exponential factorization of the map:

$$\begin{aligned} &: B_{full} : \;=\; : B_{spin} : \; : B_{orbt} : \qquad (6)\\ &: B_{orbt} : \;=\; : A_1 : exp(: F_3 :) \cdots exp(: F_{m+1} :)\\ &: B_{spin} : \;=\; : A_2 : exp(: \vec{U}_1 \cdot \vec{S} :) \cdots exp(: \vec{U}_m \cdot \vec{S} :) \end{aligned}$$

This factorization represents the map (5) with the precision $O(\mid \vec{S} \mid + \mid \vec{x} \mid^{m+1})$ in the sense that:

$$\begin{aligned} &: B_{full} : \vec{x} \;=\; \vec{F}(\vec{x}) + O(\mid \vec{S} \mid + \mid \vec{x} \mid^{m+1})\\ &: B_{full} : \vec{S} \;=\; A(\vec{x}) \cdot \vec{S} + O(\mid \vec{S} \mid + \mid \vec{x} \mid^{m+1}) \cdot \vec{S} \end{aligned}$$

Here $\vec{F}_k(\vec{x})$ and $\vec{U}_k(\vec{x})$ are homogeneous polynomials of order k.

The exponential representation is comfortable in the sense that it allows to deal not with maps but with their symbols (with algebra of Hamiltonians). It is also possible to find further types of factorizations which we will not mention here.

IV. Invariant Functions and Equilibrium Polarization Direction

The function $V(\vec{z})$ is called an invariant function of the map (4) if

$$V(\vec{z}) = V(\vec{f}(\vec{z})) \qquad (7)$$

In this section the conception of the dependent on the spin invariant functions is introduced. Taking into account that we neglect the terms proportional to $\hbar$, it is enough to consider these functions in the form:

$$V(\vec{z}) = b(\vec{x}) + \vec{g}(\vec{x}) \cdot \vec{S} \qquad (8)$$

If we substitute (8) in (7), we will see that $b(\vec{x})$ is the usual invariant function of the orbital part of the map. It means that one can find $V(\vec{z})$ only as:

$$V(\vec{z}) = \vec{g}(\vec{x}) \cdot \vec{S} \qquad (9)$$

Equation (7) for the function (9) becomes:

$$A(\vec{x}) \cdot \vec{g}(\vec{x}) = \vec{g}(\vec{F}(\vec{x})) \qquad (10)$$

From (10) it follows that

a) $\mid \vec{g}(\vec{x}) \mid^2$ is an invariant function of the map $\vec{F}(\vec{x})$.

b) If $b(\vec{x}) = b(\vec{F}(\vec{x}))$ and $V(\vec{z})$ are invariant functions then $b(\vec{x}) \cdot V(\vec{z})$ is an invariant function also. This allows to identify two invariant functions V_1 and V_2 if $V_1(\vec{z}) = b(\vec{x}) \cdot V_2(\vec{z})$ or $V_2(\vec{z}) = b(\vec{x}) \cdot V_1(\vec{z})$.

We will call the function $V(\vec{z})$ the nondegenerate one if $\mid \vec{g}(\vec{0}) \mid \neq 0$. In this case it is possible to believe that $\mid \vec{g}(\vec{x}) \mid = 1$.

The Poisson bracket of two invariant functions $V_1(\vec{z})$ and $V_2(\vec{z})$ is an invariant function $V_3(\vec{z})$ again.

$$V_3(\vec{z}) = [\vec{g}_1(\vec{x}) \times \vec{g}_2(\vec{x})] \cdot \vec{S}$$

It means that if the matrix $A(\vec{0}) \neq I$ then there is not more than one nondegenerate invariant function (with the precision up to the multiplication on the invariant function of the orbital map).

It is possible to show that if the nondegenerate invariant $V(\vec{z})$ in the form (9) exists and the fixed point of the orbital motion $\vec{x} = \vec{0}$ is stable then the vector $\vec{g}(\vec{x})$ defines the direction along which the polarization of a particle is conserved. This definition of the equilibrium polarization direction does not depend on the selection of the coordinate system and on the Hamiltonian form of the orbit motion.

In the case when the action-angle variables I, φ for the orbital motion and the Derbenev and Kondratenko vector $\vec{n}(I, \varphi)$ [3] exist, the introduced vector $\vec{g}(\vec{x})$ gives the one turn boundary conditions for $\vec{n}$.

V. Normal Form Method for Maps with Spin Variables

In this part we present an arbitrary order (in respect of the orbital variables) normal form algorithm that allows to obtain the vector $\vec{g}(\vec{x})$ in the formal power series form in the nonresonance case. This algorithm is in the Lie algebraic framework [4], [5] and uses the map factorization (6).

As usually, the algorithm consists of a sequence of coordinate transformations of the map:

$$\begin{aligned} &: C_{full} : \;=\; : C_{spin} : \; : C_{orbt} :\\ &: C_{orbt} : \;=\; exp(: K_{m+1} :) \cdots exp(: K_3 :) : C_1 :\\ &: C_{spin} : \;=\; exp(: \vec{P}_m \cdot \vec{S} :) \cdots exp(: \vec{P}_1 \cdot \vec{S} :) : C_2 : \end{aligned}$$

Here $: C_{orbt} :$ is the coordinate transformation reducing the orbital part of the map to the normal form:

$$: C_{orbt} : \; : B_{orbt} : \; : C_{orbt} :^{-1} =_m exp(: H_{orbt}(I) :)$$

where $=_m$ indicates that the right and left side agree up to order m and I, $H_{orbt}(I)$ are action variables and pseudohamiltonian of the orbital motion.

The full transformation $: C_{full} :$ reduces the full map to the normal form:

$$: C_{full} : \; : B_{full} : \; : C_{full} :^{-1} =_m$$
$$exp(: H_{orbt}(I) + \Omega_{spin}(I) \cdot S_3 :) \qquad (11)$$

where $\Omega_{spin}(I)$ is the spin tune.

Omitting the details we note that the map (11) has the nondegenerate invariant function $V = S_3$. In the initial variables this function has the form $V = : C_{full} :^{-1} \cdot S_3$.

VI. Connection between SO(3) and SU(2) Groups and One Turn Map Computation

The orthogonal 3×3 matrix $A(\vec{x}) \in SO(3)$ consists of 9 elements, but for its definition it is enough to have a smaller number of parameters. Usually this fact is used in the spinor formalism. A more classical approach is described in this paper.

In correspondence to the vector $\vec{S}$ we set the matrix

$$L = \begin{pmatrix} S_3 & S_1 + iS_2 \\ S_1 - iS_2 & -S_3 \end{pmatrix}$$

and introduce the matrix B ($B^* = -B$):

$$B = \frac{i}{2} \cdot \begin{pmatrix} W_3 & W_1 + iW_2 \\ W_1 - iW_2 & -W_3 \end{pmatrix}$$

By means of the matrices L and B one can write the spin part of motion equations (1) in form of a Lax equation:

$$\frac{dL}{d\tau} = B \cdot L - L \cdot B \qquad (12)$$

If the matrix $U(\tau)$ satisfies the equation

$$\frac{dU}{d\tau} = B \cdot U, \quad U(0) = I \qquad (13)$$

then $U(\tau) \in SU(2)$ and the solution of (12) for $L(0) = L_0$ is given by the formula:

$$L(\tau) = U(\tau) \cdot L_0 \cdot U^*(\tau) \qquad (14)$$

Thus we only need to calculate the solution of equation (13) for one turn. This approach is advantageous because the calculation with help of $SU(2)$ matrices requires a smaller number of arithmetic operations than with $SO(3)$ matrices.

VII. Implementation in Computer Code VasiLIE

All algorithms presented in this paper have been implemented up to arbitrary high order in the code VasiLIE [6]. This code allows to obtain one turn Taylor maps for orbital and spin motion, the invariant functions for the orbital motion, the equilibrium polarization direction and the orbital and spin tunes for the proton storage rings. This program is being created specially for computers like IBM PC with small memory ($1 - 2$ MB).

VIII. Acknowledgments

The authors would like to thank A.M. Kondratenko and Yu. Senichev for many useful discussions.

References

[1] V. Bargman, L. Michel, and V. L. Telegdi. *Phys. Rev. Lett.*, 2:435, 1959.

[2] A. J. Dragt and J. M. Finn. *Journal of Math. Physics*, 17:2215, 1976.

[3] Ya. S. Derbenev, A. M. Kondratenko. *Sov. Phys. JETP*, 37: No.6:968, 1973.

[4] E. Forest, M. Berz, and J. Irwin. Part. Acc., 24:91, 1989.

[5] K. Yokoya. *KEK Preprint*, 86-90.

[6] V. Balandin and N. Golubeva. *Proceedings of the XV Conf. on High Energy Particle Accelerators, Int. J. Mod. Phys. A*, 2B:998, 1992.