Remarks on the differential Luminosity in the weak disruption and the transition region

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Abstract

The time dependence of the different enhancement factors of the luminosity, in the region of weak disruption, has been discussed here, for the first time, and the corresponding time integral has been explicitely discussed upon. For the case of stronger disruption (transition region), the existing numerical results from various research works, has been checked analytically with the help of a square distribution.

1 Introduction

In order to estimate the parameters [1]..[7] of the future accelerators, especially the e^+/e^- linear colliders accelerator [8]..[11], the increment of luminosity, which results due to disruption, has to be given more consideration. In this paper, the time dependence of the differential enhancement factor (d.e.f.) of a weak disruption D, as well as that of a transition region, is discussed. For the weak disruption region the d.e.f. is expessed in terms of the Gauss' error function. For the transition region, the numerous results available [2] are been checked analytically with the help of a square distribution. The numerical results [2, 9, 11] and our analytical solution are in very good agreement.

2 Differential enhancement factor in the case of weak disruption

As a result of the penetration of the electron in the positron bunch, the respective charged particles do not move along straight but on curved paths. As it is known, this causes the pinch-effect, which then results into the increment of the luminosity. Thus the behaviour of the d.e.f.

$$\frac{dH_D}{dt} = \frac{1}{\mathcal{L}_0}\frac{d\mathcal{L}}{dt} \tag{1}$$

can be defined with the aid of

$$\begin{aligned} \mathcal{L} &= f\int\frac{dN_1 dN_2}{dA_\perp} \\ &= fN^2\int n_1(x,y,z_1,t)n_2(x,y,z_2,t)dxdydz_1dz_2 \end{aligned} \tag{2}$$

where the relationship between z_1, z_2 is $2ct = -z_1 - z_2$ [2]. The d.e.f. can be obtained from equation (2) in terms of cylindrical coordinates for (D = 0)(subscript 0) with the Gauss' radial symmetrical bunchgeometry of the d.e.f.

$$\frac{dH_{D0}}{dt} = \frac{1}{\mathcal{L}_0}\frac{df_0}{dt} = \frac{c}{\sqrt{\pi}\sigma_z}\exp\left\{-\frac{c^2t^2}{\sigma^2}\right\} \tag{3}$$

For different logitudinal and radial bunch distribution, The enhancement factor for weak disruption (D≥0) is given by:

$$H_D = 1 + \frac{1}{2}D\cdot\left\{\begin{array}{l}\frac{1}{2}rad.hom\\ \frac{2}{3}rad.gauss\end{array}\right\}\left\{\begin{array}{l}\frac{1}{\sqrt{3}}long.hom\\ \frac{1}{\sqrt{\pi}}long.gauss\end{array}\right\} \tag{4}$$

and the disruption parameter is

$$D = \frac{r_e\sigma_z N}{\gamma\sigma_0}; \quad r_e = \frac{\hbar c\alpha_s}{m_0c^2} \tag{5}$$

In addition to the above known results, a new d.e.f. is being introduced.

$$\frac{dH_D}{dt} = \frac{1}{\mathcal{L}_0}\frac{d\mathcal{L}_0}{dt} + \frac{2c^2(2\pi)^3 r_e N}{\gamma}V\tilde{U}(t) \tag{6}$$

where

$$V = \frac{\int_0^{+\infty} drn_r^3(r)}{\int_0^{+\infty} drn_r^2(r)} = \left\{\begin{array}{ll}\frac{1}{2}\frac{1}{\sigma_0^2} & hom.\\ \frac{2}{3}\frac{1}{\sigma_r^2} & gauss.\end{array}\right\} \tag{7}$$

is the radial bunchgeometry and

$$\tilde{U} = \int_0^{+\infty} dz_1 n_z(z_1)n_z(-z-2ct)\left[g(t,z_1)+g(t,-2ct-z_1)\right] \tag{8}$$

the longitudinal bunch geometry with respect to dH_0/dt. The functions g in (8) are defined as

$$g(t,z_j) = \frac{1}{4c^2}\int_0^{\infty} d\tau\, \tau n_z(\tau+z_k) \qquad \begin{array}{l} j=1\Rightarrow k=2\\ j=2\Rightarrow k=1\end{array} \tag{9}$$

The equation (6) can be further formulated for the longitudinal Gauss' distribution:

$$n_z(z) = \frac{1}{(2\pi)^{\frac{3}{2}}\sigma_z}\exp\left\{-\frac{z^2}{2\sigma_z^2}\right\} \tag{10}$$

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as follows

$$\tilde{U}(t) =$$

$$= \int_{-\infty}^{+\infty} dz_1 n_z(z_1) n_z(z_2) \frac{1}{4c^2} \times$$

$$\times \left[\int_0^{+\infty} d\tau\, \tau \{ n_z(\tau + z_2) + n_z(\tau + z_1) \} \right]$$

$$= \frac{1}{4c^2} \int_{-\infty}^{+\infty} dz_1 n_z(z_1) n_z(-2ct - z_1) \times$$

$$\times \left[\int_0^{+\infty} d\tau\, \tau \{ n_z(\tau + (-2ct) - z_1) + n_z(\tau + z_1) \} \right]$$

$$= \frac{1}{4c^2} \frac{1}{\left[2\pi\sqrt{2\pi}\sigma_z\right]} \int_{-\infty}^{+\infty} dz_1 e^{-\frac{z_1^2}{2\sigma_z^2}} e^{-\frac{(2ct+z_1)^2}{2\sigma_z^2}} \times$$

$$\times \int_0^{+\infty} d\tau\, \tau \left\{ e^{-\frac{(\tau-2ct-z_1)^2}{2\sigma_z^2}} + e^{+\frac{(\tau+z_1)^2}{2\sigma_z^2}} \right\} \qquad (11)$$

After various substitutions and numerous mathematical steps (not discussed here) the following equation results

$$\tilde{U}(t) = 2\sqrt{\frac{2\pi\sigma_z^2}{3}} \frac{1}{4c^2} \frac{1}{\left(2\pi\sqrt{2\pi}\sigma_z\right)^3} e^{-\frac{-c^2t^2}{\sigma_z^2}} \times \qquad (12)$$

$$\times \left\{ \frac{3\sigma_z^2}{2} e^{-\frac{-c^2t^2}{\sigma_z^2}} + ct\sigma_z\sqrt{3}\frac{\sqrt{\pi}}{2} \left[1 - erf\left\{ \frac{ct}{\sqrt{3}\sigma_z} \right\} \right] \right\}$$

From this equation a new expression for dH_0/dt can be dreived:

$$\frac{dH_D}{dt} = \qquad (13)$$

$$= \frac{c}{\sqrt{\pi}\sigma_z} e^{-\frac{-c^2t^2}{\sigma_z^2}} + \frac{cr_eN\left[\frac{1}{2}\int_0^{\infty} sdrn_r^3(r)\right]}{\gamma\left[\int_0^{\infty} sdrn_r^2(r)\right]} \frac{\sqrt{3}}{2\pi} \times$$

$$\times e^{-\frac{-c^2t^2}{\sigma_z^2}} \left[e^{-\frac{-c^2t^2}{3\sigma_z^2}} + \frac{ct\sqrt{\pi}}{\sqrt{3}\sigma_z} \left\{ 1 - erf\left(\frac{ct}{\sqrt{3}\sigma_z} \right) \right\} \right]$$

$$= \frac{c}{\sqrt{\pi}\sigma_z} e^{-\frac{-c^2t^2}{\sigma_z^2}} + \frac{\sqrt{3}}{2\pi} \frac{cr_eN}{\gamma} V \frac{1}{2} \times$$

$$\times e^{-\frac{-c^2t^2}{\sigma_z^2}} \left[e^{-\frac{-c^2t^2}{3\sigma_z^2}} + \frac{ct\sqrt{\pi}}{\sqrt{3}\sigma_z} \left\{ 1 - erf\left(\frac{ct}{\sqrt{3}\sigma_z} \right) \right\} \right]$$

3 Analytical results for the transition region

The characteristics of trajectories and those of the density distribution (under the aspect of longitudinal Gauss' distribution), in case of larger disruption parameter ($1 \leq D \leq 10$) can be obtained only numerically. The numerical results mentioned above shall now be proven with the aid of equivalent longitudinal square distribution. The general differential equation [9, 11, 12] for a radial motion is

$$\ddot{r}(t) = \frac{c^2}{c^2 m_0 \gamma}(-4\pi\hbar c\alpha_s N)\frac{1}{r(t)} \int_0^{r(t)} dr^* \, r^* n(t, r^*, z) \qquad (14)$$

Considering only the paraxial beams, the Taylor's series can be written as:

$$\ddot{r}(t) = -\frac{4\pi c^2 N r_e}{\gamma} \frac{1}{2} n(t, 0, z) r(t) \qquad (15)$$

for which the information regarding the density distribution along the beam axis ($r = 0$) [2, 9, 11, 12] is necessary. This can be expressed as

$$n(t, 0, z) = \qquad (16)$$

$$\frac{1}{(2\pi)^{3/2}\sigma_z} \frac{e^{-\frac{(2ct+z_1)^2}{2\sigma_z^2}}}{\sigma_z^2} \left[1 + \frac{4\pi c^2 N r_e^2}{\gamma\sigma_r^2} g(t, z_1) \right]$$

(15) and (16) lead to the numerical and analytical solutions mentioned in the references [2].

(i) The slope on the focal point (in equation (16) in a first approxiation only the first term "1" is considered):

$$\dot{r}(\tilde{t}_0) = -\frac{3}{4}\frac{c}{\sigma_z}\sqrt{\frac{D}{2}} \qquad (17)$$

(ii) The focal point t0 within the limits D $\in [1, 10]$ is

$$\tilde{t}_0 \approx 2\frac{\sigma_z}{cD} \qquad (18)$$

In the differential equation (15) the longitudinal Gauss' distribution

$$n_z(z) = \frac{1}{2\pi} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_z} \exp\left(-\frac{z^2}{2\sigma_z^2} \right) \qquad (19)$$

is substituted by the areal equivalent quadratic distribution

$$n_z(z) = \begin{cases} \frac{1}{2\pi l} & ; |z_z| \leq \frac{l}{2} \\ 0 & ; |z_z| > \frac{l}{2} \end{cases} \qquad (20)$$

Neglecting the second term in (16) the differential equation (15)

$$\frac{d^2 r(\tilde{t})}{d\tilde{t}} = -\frac{2\pi}{2\pi} \frac{c^2}{\sqrt{2\pi}} \frac{D}{\sigma_z^2} e^{-\frac{-2c^2t^2}{\sigma_z^2}} \cdot r(\tilde{t}) \qquad (21)$$

turns into $\Downarrow$

$$\frac{d^2 r(\tilde{t})}{d\tilde{t}} = -\frac{2\pi}{2\pi} \frac{c^2}{\sqrt{2\pi}} \frac{D}{\sigma_z^2} e^{-\frac{-2c^2t^2}{\sigma_z^2}} \cdot r(\tilde{t}) \qquad (22)$$

where $z_2 = -2ct - z_1 = 2c\tilde{t}$. From equation (22) we obtain for $|\tilde{t}| \leq L/4c$ a simple differential equation with harmonic solutions

$$\begin{aligned} r(\tilde{t}) &= A_m \sin(W\tilde{t}) + B_m \cos(W\tilde{t}) \\ \dot{r}(\tilde{t}) &= A_m W \cos(W\tilde{t}) - B_m W \sin(W\tilde{t}) \\ W^2 &:= -2\pi c^2 \frac{D}{\sigma_z} \cdot \begin{cases} \frac{1}{2\pi l} & ; |\tilde{t}| \leq \frac{l}{4c} \\ 0 & ; else \end{cases} \end{aligned} \tag{23}$$

The solutions have to be differentiable and must be matched to the the general solution for $|\tilde{t}| \geq L/4c$

$$\begin{aligned} r(\tilde{t}) = A_l + B_l \cdot \tilde{t} = 1 + 0 \cdot \tilde{t} \quad &, \dot{r}(\tilde{t}) = 0 \\ r(\tilde{t}) = A_R + B_R \cdot \tilde{t} \quad &, \dot{r}(\tilde{t}) = B_R \end{aligned} \tag{24}$$

Due to the above condition, and $r(t_0) = 0$ the focal point can be written as

$$\tan(W\tilde{t}_0) = \frac{1}{tan\left(W\frac{l}{4c}\right)} \tag{25}$$

and the slope $r(\tilde{t}_0)$ as

$$\dot{r}(\tilde{t}_0) = -W \left[\sin\left(W\frac{l}{4c}\right) \frac{tan\left(W\frac{l}{4c}\right)}{\sqrt{\tan^2\left(W\frac{l}{4c}\right) + 1}} + \right. \\ \left. + \cos\left(W\frac{l}{4c}\right) \frac{1}{\tan^2\left(W\frac{l}{4c}\right) + 1} \right] \tag{26}$$

Hence the final relation can be given as:

$$\dot{r}(\tilde{t}) = -W = -\frac{c}{\sqrt{2\sqrt{3}}} \frac{\sqrt{D}}{\sigma_z} \tag{27}$$

This matches amazingly with the numerical results already mentioned above.

$$\frac{1}{\sqrt{\sqrt{3}}} = \frac{1}{\sqrt[4]{3}} = 0,75983 \approx \frac{3}{4} \tag{28}$$

The equation (26) can lead to a corresponding result for the region {3; 6,6}, after various series expansions as

$$\tilde{t}_0 \approx 2,4 \frac{\sigma_z}{CD} \tag{29}$$

4 Conclusion

For the different bunch geometries of the d.e.f in the weak focussing region a time dependency has been established and relation has been derived. Besides, the various results from different references have been evaluated. For the transition region an extremely good match has been obtained between the numerical methods achieved from a series complex mathematical operations and the results quoted. This was possible due to the substitution of longitudinal Gauss' distribution through an equivalent square distribution. This simplified the differential equation upon which the analytical methods could be easily applied.

References

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