Acceleration and Bunching by a Gap

S. Kulinski
INFN, LNF, P.O. Box 13, 00044 Frascati (Italy)

Abstract

The relativistic equations of axial motion of a charged particle in an RF electric field inside a gap are analysed.The solution to these equations can be expressed as the incomplete elliptic integrals of the first and third kinds. The approximate solutions based on the average particle velocity in the gap are also found. The bunching by a gap is analysed by considering the phase ϕ_e after the drift as a function of the entrance phase ϕ_i: $\phi_e=\phi_i+\phi_t+\phi_d$, ϕ_t is the transit angle in the gap, ϕ_d drift angle. Starting from the general condition for bunching $d\phi_e/d\phi_i \leq 1$ it is shown that bunching begins with the input phase ϕ_{ipm} corresponding to the minimum of the momentum p_m at the exit of the gap and ends at phase ϕ_{ipM} when the momentum attains its maximum p_M. The width of the bunched phases $\Delta\phi_e$ is analysed as a function of different parameters and can be optimized according to the imposed conditions.

I. INTRODUCTION

Bunching is the basis of many microwave devices in which the interaction between the beam of particles and an electromagnetic wave plays an essential role as for instance in klystrons or linear accelerators. Consequently the problem of bunching was analysed by many authors [1-3]. Usually this analysis is based on some simplifying assumptions. The most important are: i) The transit time of a particle through the interaction gap is negligible in comparison with the field period. ii) The AC voltage V_g in a gap is small in comparison with the DC voltage V_0. iii) Space charge effects are ignored.

In the present paper the analysis will be made without the first two of above restrictions. To take properly into account these effects the relativistic equations of motion of particles in a gap are solved.Spatially constant field in a gap is assumed, since then the analytical solutions are possible. However, as it was shown by numerical calculations [3], the results for other field distributions e.g. gaussian are similar. The analysis is made for electrons, but the results can be used also for other charged particles.

II. AXIAL MOTION IN A GAP

A. Equations of Motion

Equations of motion are

$$d\gamma/ds = A \cos\phi \qquad (1)$$

$$d\phi/ds = 2\pi\gamma/(\gamma^2 - 1)^{1/2} \qquad (2)$$

where, γ is the relativistic energy factor, $s = z/\lambda$, z-axial distance, λ- wave-length, ϕ- electric field phase in a gap, $A = q\lambda E/W_o$, q-charge, E- electric field intensity, $W_o = m_oc^2$- particle rest energy.

In the case of E = const. an analytical solution to Eqs (1) and (2) can be found. First by elimination of ds we find equation for the normalized momentum $p=mv/m_oc=\gamma\beta$, ($\beta=v/c$)

$$p = p_o + A_1 (\sin\phi - \sin\phi_o) = \gamma\beta \qquad (3)$$

Here subscript o denotes the initial values. Eq. (3) defines the phase trajectories in the phase space (p,ϕ), however, it does not give the dependences $\gamma = \gamma(s)$ and $\phi = \phi(s)$. To obtain these relations using Eq.(3) we express γ and $\cos\phi$ as functions of momentum p. Inserting these relations into Eq. (1) and integrating we get

$$s = \pm\frac{1}{2\pi}\int_{p_o}^{p}\frac{pdp}{\sqrt{(1+p^2)(A_1^2-(p-p_o+A_1\sin\phi_o)^2)}} \qquad (4)$$

The integral (4) can be expressed in terms of the incomplete elliptic integrals of the first and third order [3]. Since the expressions with these integrals are rather complicated, for numerical calculations it could be preferable to integrate directly Eqs. (1) and (2) or (4). However, it would be more effective if one could solve analytically, even approximately, Eqs (1,2) without the necessity to calculate the integral (4).

B. Approximate Solutions to the Equations

The phase ϕ as seen by a particle can be written as $\phi = \phi_o + \phi_t(s)$, where ϕ_o is the initial phase and $\phi_t(s)$ is the transit phase given by

$$\phi_t(s) = 2\pi\int_{s_o}^{s}\frac{ds}{\beta(s)} = \frac{2\pi(s-s_o)}{\beta_{av}} \qquad (5)$$

β_{av} is the average particle velocity in the gap as defined by (5). To find β_{av} we will average the Eq. (3) for p over the changes of phase ($\phi_o,\phi_o + \phi_t(s)$)

$$p_{av} = p_o + A_1((\sin\phi)_{av} - \sin\phi_o) = 1/(1/\beta_{av}^2-1)^{1/2} \qquad (6)$$

where

$$(\sin\phi)_{av}=1/\phi_t\int_{\phi_o}^{\phi_o+\phi_t}\sin\phi d\phi = (\cos\phi_o - \cos(\phi_o+\phi_t))/\phi_t \qquad (7)$$

Inserting (7) into (6) we will get equation for β_{av}

$$p_o - (1/(\beta_{av})^2 - 1)^{-1/2} = A_1(\sin\phi_o - (\cos\phi_o - \cos(\phi_o+\phi_t))/\phi_t) \qquad (8)$$

In the case of small changes of the velocity in a gap $\Delta\beta/\beta \ll 1$ equation (8) can be reduced to quadratic equation for the small quantity proportional to $\Delta\beta/\beta$. To obtain this equation we put

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$x_o = 1/\beta_o$, $x = 1/\beta_{av} = x_o(1 + y)$, $\phi_t = 2\pi sx = \phi_{to}(1 + y)$, $\phi_{to} = 2\pi s x_o$ is the transit angle of the particle with the input velocity β_o. Assuming that $y << 1$, valid surely for small signal bunching, we can expand functions of Eq.(8) into series up to the second order in y in the vicinity of x_o to obtain quadratic eqution for y with the aid of which β_{av} is found.

III. BUNCHING

A. Central Particle

In the process of bunching by the velocity modulation a special role plays the central particle. Usually this is the particle which passes an interaction gap without the change of energy. We should then find such a phase ϕ_o for which a particle leaves the gap without changing its momentum p_o. Using Eq. (3) we obtain the obvious relation

$$\sin\phi_{oc} = \sin(\phi_{oc} + \phi_{tc}) = \sin(\phi - \phi_{oc}) \qquad (9)$$

and

$$\phi_{oc} = (k + 1/2) \pi - 0.5 \phi_{tc} \qquad (10)$$

where ϕ_{tc} is the transit angle of the central particle. Assuming that $\phi_{tc} < \pi$ (narrow gap not to large signals) we will have for electrons $k = 0$ and $\phi_{oc} = (\pi - \phi_{tc})/2$.

The precise solution for ϕ_{oc} can be now obtained with the aid of Eq. (4) where the limits of the integral should be (p_{min}, p_o) and pmin is given by (for electrons $A_1 < 0$)

$$p_{min} = p_o + A_1(1 - \sin\phi_o) \qquad (11)$$

The approximate solutions can be obtained from Eq. (8) inserting , according to Eq. (10), $\phi_o = \phi_{oc} = (\pi - \phi_{tc})/2$ to get

$$p_o - (x^2 - 1)^{-1/2} + A_1 (\sin\alpha/\alpha - \cos\alpha) \qquad (12)$$

where $x = 1/\beta_{av}$, $\alpha = \phi_{tc}/2 = \pi s x$

B. General Relations for Bunching

In the case of a bunching system composed of a gap of length L and a drift D, the exit angle is given by

$$\phi_e = \phi_i + \phi_t + \phi_d = \phi_i + F/\beta \qquad (13)$$

where ϕ_i -input phase, $\phi_t = 2 \pi L/(\lambda \beta_{av})$ is the transit angle in the gap, $\phi_d = 2\pi D/(\lambda \beta)$ is the drift transit angle and β is the exit velocity. Putting $\beta_{av} = h\beta$, where h is usually close to 1, F is equal to

$$F = 2 \pi D/\lambda (1 + L/hD) \qquad (14)$$

General condition for bunching is

$$d\phi_e/d\phi_i \le 1 \qquad (15)$$

Taking into acount (13) and (14) Eq. (15) becomes

$$d\phi_e/d\phi_i = 1 - F/(p^2(1 + p^2)^{1/2}) dp/d\phi_i \le 1 \qquad (16)$$

We begin with some general conditions for bunching, which can be obtained from the relation (16). Generally, since $F > 0$ then the condition (16) requires that $dp/d\phi_i \ge 0$. It means that bunching starts with phase $\phi_i = \phi_{ipm}$ corresponding to p_{min} at the exit and ends with the phase ϕ_{ipM} for which $p=p_{max}$. The range of bunched input phases is

$$\Delta\phi_i = \phi_{ipM} - \phi_{ipm} \qquad (17)$$

Then to define the starting conditions for bunching we should find the extrema of the momentum p as a function of ϕ_i. For $\Delta\phi_e$ we should find extrema of ϕ_e using the equation $d\phi_e/d\phi_i = 0$. Generally, depending upon the parameter F and momentum p_o this equation can have 0, 1 or two solutions. In the case of 0 or one solution we have $0 \le d\phi_e/d\phi_i \le 1$. The bunched output phase ϕ_e is a monotonic function of the input phase ϕ_i. $\Delta\phi_e$ is equal to $\Delta\phi_e = \phi_{epM} - \phi_{epm}$, where ϕ_{epM} and ϕ_{epm} are the output phases after the drift corresponding to p_{max} and p_{min} at the exit of the gap.

In the case of two solutions we have four characteristic points on the curve $\phi_e = \phi_e(\phi_i)$ on which depends the width $\Delta\phi_e$. The bunching starts with $p = p_m$ and $\phi_i = \phi_{ipm}$. For $\phi_i > \phi_{ipm}$ $dp/d\phi_i > 0$ and $d\phi_e/d\phi_i > 0$ up to the point where $d\phi_e/d\phi_i = 0$ and $\phi_e = \phi_{eM}$. Beyond this point $dp/d\phi_i > 0$ but $d\phi_e/d\phi_i < 0$ until the point where $d\phi_e/d\phi_i = 0$ and $\phi_e = \phi_{em}$ is reached. After that ϕ_e begins to increase again and bunching stops when $dp/d\phi_i = 0$, $p = p_M$ and $\phi_e = \phi_{epM}$. The width $\Delta\phi_e$ is equal now

$$\Delta\phi_e = \max (\phi_{eM}, \phi_{epM}) - \min (\phi_{em}, \phi_{epm}) \qquad (18)$$

Further analysis will be possible when the extrema of the momentum p and the extrema of ϕ_e are found.

C. Extrema of the Momentum P

At the end of the gap the momentum p is given by

$$p = p_o + A_1 (\sin(\phi_i + \phi_t) - \phi_i) \qquad (19)$$

It can be shown that $dp/d\phi_i$ is equal to

$$dp/d\phi_i = \frac{p^2\sqrt{1+p^2}A_1(\cos(\phi_i+\phi_t)-\cos\phi_i)}{p^2\sqrt{1+p^2}+A_1 g\cos(\phi_i+\phi_t)} \qquad (20)$$

Here $g = 2 \pi L/(h \lambda)$. Then from $dp/d\phi_i = 0$ follows

$$\cos(\phi_i + \phi_t) - \cos\phi_i = 0 \qquad (21)$$

Eq. (21) has two principal solution in the range $(-\pi, \pi)$

$$\phi_i = -\alpha, \qquad p_{extr} = p_o + 2\,A_1 \sin\alpha$$
$$\phi_i = \pi - \alpha, \qquad p_{extr} = p_o - 2\,A_1 \sin\alpha \tag{22}$$

where $\alpha = \phi_t/2$. For electrons $A_1 < 0$ and the first solution is for p_{min} the second for p_{max}. The range of bunched input phases is

$$\Delta\phi i = \phi_{ipM} - \phi_{ipm} = \pi + \alpha_{ipm} - \alpha_{ipM} \tag{23}$$

Since $\alpha_{ipm} > \alpha_{ipM}$ then $\Delta\phi i > \pi$. Usually for small signal bunching $p_o >> 2\,A_1$, the difference between p_M and p_m is small so that $\Delta\phi i$ is close to π. For large signal, when A_1 can be comparable with p_o $\Delta\phi_i$ can substatially differ from π. To solve completely the problem we should find the transit angles ϕ_{tpm} and ϕ_{tpM}. The precise solution can be obtained with the aid of equation (4). Approximate solution is obtained inserting relations (22) and (23) into (8) to get the equation for $x = 1/\beta_{av}$

$$p_o - (x^2 - 1)^{-1/2} + A_1 \sin\pi s x = 0 \tag{24}$$

D. Extrema of the Bunched Output Phase Φ_e

The extrema of ϕ_e are given by $d\phi_e/d\phi_i = 0$. Using (16) and (20) we get

$$p^2(1 + p^2)^{1/2} \text{-} (2\pi D/\lambda)A_1(\cos(\phi_i + \phi_t)\text{-}(1 + L/hD)\cos\phi_i) = 0 \tag{25}$$

In this equation the transit angle ϕ_t is a function of ϕ_i so that in principle it is an equation for ϕi. However, since we do not know explicitely the functional dependence $\phi_t = f(\phi_i)$, we should use some iterative procedure to solve Eq. (25). For a given value of ϕ_i, ϕ_t can be found either precisely with the aid of Eq. (4) or approximately using equation (8). Usually for small signal bunching the simpler approximate solution is sufficiently accurate. Generally, as it was already mentioned above Eq. (25) can have 0, 1, or 2 solution for ϕ_i in the range $(-\pi, \pi)$. The number of solutions depends on the parameters of the bunching system: Vo, Vb, D, L and λ. Solution of Eq. (25) defines the type of bunching and $\Delta\phi_e$. This, together with previously found quantities : $\Delta\phi_i$, p_{min}, p_{max} defines completely the bunching system giving not only the effectiveness of phase bunching $\Delta\phi_i/\Delta\phi_e$, but also the energy dispersion introduced by the system $\Delta p = p_{max} - p_{min}$.

Up to now we used a general definition of the bunching factor $R = \Delta\phi_i/\Delta\phi_e$ with $\Delta\phi_i$ and $\Delta\phi_e$ defined by Eqs (17) and (18) correspondingly. However, in practice e.g. for positron production or injection into superconducting cavities, where both small phase and energy dispersion is essential, we often would like to have rather narrow well bunched output phases. It means that in the vicinity of the central particle the changes of ϕ_e should be small for sufficiently large variations of ϕ_i. This can be done by choosing the bunching parameter in such a way as to have two extrema of ϕ_e close to each other, since then $d\phi_e/d\phi_i \approx 0$, variations of ϕ_e are small and given by

$$\phi_e \approx \phi_{ee} + 0.5\,(\phi_i - \phi_{ie})^2\, d^2\phi_e/d\phi_i^2 \tag{26}$$

where ϕ_{ee} corresponds to the extremum of ϕ_e and ϕ_{ie} the value of ϕ_i in this point. We can now define the effective value of bunching as

$$R_{eff} = (\phi_{iM} - \phi_{im})/(\phi_{eM} - \phi_{em}) \tag{27}$$

where ϕ_{eM} and ϕ_{em} are the extrema of ϕ_e, ϕ_{iM} and ϕ_{im} are the values of the input phase ϕ_i corresponding to the points in which ϕ_e is equal to ϕ_{em} and ϕ_{eM}, outside the points of extrema of ϕ_e. These values can be found from Eq. (26) . The second derivative $d^2\phi e/d\phi_i^2$ is found by differetiating once again Eq. (16)

C. Numerical Example

The program GAPAC (GAP ACceleration) has been written to make the numerical calculations and to check the validity of approximations. As an example we have chosen the prebuncher made for the SC accelerator LISA of Frascati Laboratories.The main parameters of this prebuncher are: V_o = 100 kV, λ = 0.6m, gap length L = 0.1 λ = 0.06m, drift length D = 1.36m, gap RF voltage $V_g \approx$ 10kV. Two kinds of calculations have been made. First the value of classical bunching parameter $B_{po} = D\, V_g\, \pi/(V_o\, \lambda\, \beta_o)$ was found for which $R = \Delta\phi_i/\Delta\phi_e$ is maximum. $\Delta\phi_i$ and $\Delta\phi_e$ are given by Eqs. (17) and (18) correspondingly. The calculation has shown that the maximum is $R_M \approx 7$ and is obtained for $B_{po} \approx 1.81$. In fact these values seems to be common for small signal bunching. In the second case we were looking for the solution giving $\Delta\phi_i$ greater than 60 degrees and $\Delta\phi_e$ of the order of one degree. The solution chosen was B_{po} =1.45, $\Delta\phi_i \approx 82°$, $\Delta\phi_e = \phi_{eM} - \phi_{em} \approx 1.2°$ and $R_{eff} \approx 70.5$. Theoretically there was no problem to get $Re\phi\phi$ an order of magnitude larger e.g. $\Delta\phi_i \approx 45°$ and $\Delta\phi_e \approx 0.1°$.

In both cases the difference between the precise and approximate solutions was below 1%. Only in the case when the extrema of ϕ_e were very close to each other the precise solution of Eq. (25) was necessary.

IV. REFERENCES

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