Matrix Nonlinear Beam Dynamics in Curvilinear Space-Time

Alexander Dymnikov
Institute of Computational Mathematics
University of St. Petersburg
Stary Petergof 198904
St. PETERSBURG, Russia

Ragnar Hellborg
Department of Physics
University of Lund
Sölvegatan 14
S-223 62 LUND, Sweden

Abstract

A general relativistic matrix theory of charged particle beam motion along an arbitrary curved optical axis in 4-space-time has been developed. This theory uses three basic matrix functions: the reference frame matrix, the curvature matrix and the electromagnetic matrix. The Cartan method of the moving 3-vector is generalized as the method of the moving 4×4 reference matrix. The curvature matrix function consists of the normal curvature, the geodesic curvature and torsion and three components of the gravitational force acting on the reference particle. The matrix equations of the beam motion and of the electromagnetic field are written.

The nonlinear equations in phase space are reformulated as linear equations in phase moment space. A new compact recursive method is proposed for integrating these linear equations. Using this method the phase volume of the beam will be strictly conserved in each step of the numerical integration.

1 Introduction

A general relativistic theory of charged particle beam motion along a curved optical axis, including the gravitational field, is important for designers of optimal beam control systems. Some basic publications on this topics can be found in [1] – [4]. In these references the scalar and tensor methods are used and usually a nonrelativistic theory is developed. In this paper a new matrix approach is presented which is based on some previous papers [5] – [7]. This approach gives the possibility to develope a matrix relativistic theory of charged particle beam motion in the most general case of curved reference trajectory, including the gravitational force and space charge. Two detailed reports on this purpose have just recently been published [8]. Some earlier papers have been devoted to the special problem of the effect of space charge. Both for the general case [9] and for a special case with an infinitely long beam and with an elliptical beam cross-section in a static electromagnetic field [10] and [11].

2 The equations of a particle beam motion and the equations of an electromagnetic field.

We understand the motion of any material body as a motion relative to other material bodies. The motion of any particle Q of the beam is described in the form of a motion relative to a single particle M of the beam. The particle M is called *the reference particle*. The trajectory of the reference particle is called *the reference trajectory* or *the optical axis* of the beam. We assume that the reference trajectory is known and the motion of the reference particle is described relative to any material body. The position of an arbitrary particle Q in the moving reference frame is determined by the 4-vector $L = \tilde{e}x$, where e is the reference frame matrix, attached to the reference particle M, x_1 and x_2 are the transverse coordinates, x_3 is the longitudinal coordinate and $x_4 = cst$. All quantities used are either dimensionless or expressed in terms of units of length of the beam motion or of the inverse length.

The complete motion of an arbitrary particle Q can be decomposed in the following way:

$$dQ = dM + dL = \tilde{e}dz_m + d(\tilde{e}x)$$

The motion of the reference frame is determined by the equation:

$$\frac{de}{ds} = e' = P(k,l)e, \; e(s_0) = e_0$$

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where:

$$P(k,l) = \begin{pmatrix} 0 & k_3 & -k_2 & l_1 \\ -k_3 & 0 & k_1 & l_2 \\ k_2 & -k_1 & 0 & l_3 \\ l_1 & l_2 & l_3 & 0 \end{pmatrix}$$

$$e\tilde{e} = G = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For the Darbou reference frame k_1 is the normal curvature, k_2 is the geodesic curvature, k_3 is the geodesic torsion. The 3-vector l is the vector corresponding to the gravitational force F, acting on the reference particle, *i.e.* $F = -pl$. The coordinates z_m are chosen such that $z_{m1} = z_{m2} = 0$, $z_{m3} = s$, $z_{m4} = ct_m$, $ds = pd\tau_m$, $dz_{m4} = \gamma d\tau_m$, $d\tau_m^2 = d\tilde{M}dM$, where p is the reference momentum and γ is the reference energy. We assume that the observer is located in the plane $x_3 = 0$, *i.e.* that all particles Q reaching this plane at different times are detected. We will study the dynamics of particles in the plane $x_3 = 0$, and therefore we have four independent variables x_1, x_2, x_4 and s.

The equations of motion for an arbitrary particle Q can be written as:

$$\left(\frac{z'}{\tau'}\right)' = \frac{1}{\tau'} P\left(\hat{B}, \hat{E}\right) z' \qquad (1)$$

where:

$$\tau' = \sqrt{\tilde{z}' G z'},\ \hat{B} = B\tau' + k,\ \hat{E} = E\tau' - l$$

$$z' = z'_m + x' + \tilde{P}(k,l)\, x$$

$$z'_{m1} = z'_{m2} = 0,\ z'_{m3} = 1,\ z'_{m4} = \frac{\gamma}{p}$$

$$z_{m4} = \int_{s_0}^{s} \frac{\gamma}{p} ds + z_{m40}$$

The fields B and E are functions of x_1, x_2, x_4 and s. The 3-vectors k and l and the scalars γ and p are functions of s.

The Maxwell equations of electromagnetic field may be written in the following matrix form:

$$\underset{\uparrow}{P}(B,E)\, G \underset{\downarrow}{\nabla}(z) + \left[P(B,E)\, GP(k,l)\right] \frac{1}{a_3} i(3) +$$

$$+ \left[P(-k,l)\, P(B,E)\, G\right] \frac{1}{a_3} i(3) + \rho \frac{dz}{d\tau} = 0 \qquad (2)$$

$$\underset{\uparrow}{P}(-E,B)\, G \underset{\downarrow}{\nabla}(z) + \left[P(-E,B)\, GP(k,l)\right] \frac{1}{a_3} i(3) +$$

$$+ \left[P(-k,l)\, P(-E,B)\, G\right] \frac{1}{a_3} i(3) = 0$$

where:

$$\tilde{i}(3) = (0,0,1,0),\ \tilde{\nabla}(z) = (\nabla_1(z) \ldots \nabla_4(z))$$

$$\nabla_i(z) = \frac{\partial}{\partial z_i},\ \nabla_i(z) = \nabla_i(x),\ i = 1,2,4$$

$$\nabla_3(z) = -\frac{a_1}{a_3}\nabla_1(x) - \frac{a_2}{a_3}\nabla_2(x) + \frac{1}{a_3}\nabla(3) - \frac{a_4}{a_3}\nabla_4(x)$$

$$\nabla(s) = \frac{\partial}{\partial s}$$

$$a_1 = -k_3 x_2 + l_1 x_4,\ a_2 = k_3 x_1 + l_2 x_4$$

$$a_3 = 1 - k_2 x_1 + k_1 x_2 + l_3 x_4,\ a_4 = \frac{\gamma}{p} + l_1 x_1 + l_2 x_2$$

3 THE METHOD OF EMBEDDING IN PHASE-MOMENT SPACE FOR SOLVING THE NONLINEAR EQUATIONS OF THE MOTION OF A PARTICLE BEAM

The analysis and calculation of the nonlinear systems of equations for beam formation are considerably simplified by a transformation from the nonlinear differential equations of motion in the phase space (x, x') to the system of linear equations in extended phase space, the phase moment space. This is the essence of the method of embedding in phase moment space. In this method the ideas that were originally presented in ref. [12] have been developed further in ref. [7].

We define recursively the rth power of the vector x as:

$$x^r = \begin{pmatrix} x_1 & \ldots & x^{r-1}(1) \\ \ldots & & \ldots \\ x_n & \ldots & x^{r-1}(n) \end{pmatrix},\ x^l(j) = \begin{pmatrix} x_j x^{l-1}(j) \\ \ldots \\ x_n x^{l-1}(n) \end{pmatrix}$$

$$x^0(j) = 1,\ j = 1,\ldots,n,\ l = 1,\ldots,r$$

This is called the r-moment of the vector x or in short, the r-moment, which has $\alpha(n,r)$ scalar elements, where:

$$\alpha(n,r) = \frac{(n-1+r)!}{(n-1)!\,r!}$$

Let us introduce the 6-vector h, given by:

$$h_i = x_i,\ i = 1,2,3,\ h_4 = \frac{p}{p(0)} x'_1$$

$$h_5 = \frac{p}{p(0)} x'_2,\ h_6 = \frac{p^3}{p^3(0)} x'_4$$

It is possible to show that in the phase space $\{h\}$ the phase volume remains constant during the motion.

Using the Taylor expansion of the functions $B(h)$, $E(h)$ and $\tau'(h)$ to transform them into a finite series, the equations of motion can be written as:

$$h' = F(s)\langle h^r\rangle \tag{3}$$

where the vector is given from $\left\langle \tilde{h}^r \right\rangle = (h^1, h^2, ..., h^r)$.

The equations for the phase moments h^s, where $s = 2, ..., r$, can be obtained with the same accuracy as eq. (3). Therefore it is possible to write the linear equation for the vector $\langle h^r\rangle$ in the following way:

$$\langle h^r\rangle' = P(s)\langle h^r\rangle \tag{4}$$

We call the matrix $P(s)$ the coefficient matrix. The solution of the linear equation for h, given by eq. (4), coincides with the solution of eq. (1), which has been obtained by the successive-approximation method. The method of reducing eq. (3) to the form of eq. (4) is reffered to as the method of embedding in phase moment space.

The solution of eq. (4) is written in terms of a matrizant in the form:

$$\langle h^r\rangle = \begin{pmatrix} h^1 \\ ... \\ h^r \end{pmatrix} = \begin{pmatrix} R^{11} & ... & ... & R^{1r} \\ 0 & R^{22} & ... & R^{2r} \\ ... & ... & ... & ... \\ 0 & 0 & ... & R^{rr} \end{pmatrix} \begin{pmatrix} h^1 \\ ... \\ h^r \end{pmatrix} =$$

$$= R(P, s/s_0)\langle h_0^r\rangle$$

The matrizant R has, in the same way as the coefficient matrix P, the form of an upper triangular block.

A continuous generalized analogue of the Gauss brackets [6] and [13] can be used to calculate the matrizant for an arbitrary coefficient matrix. In this method there is rigorous conservation of the phase volume of the beam at each stage of the calculations. An effective computer code, based on this method, has been written for studying the beam dynamics for an arbitrary axial distribution through fifth order in the nonlinearity [14].

4 Summary

A new matrix and recursive approach has been outlined for treating nonlinear optics of charged particle beams. This approach is a new analytical and computational tool for designers of optimal beam control systems. This relies on three basic matrix functions: the reference frame matrix, the curvature matrix and the electromagnetic matrix. The nonlinear equations in phase space are reformulated as linear equations in phase moment space. A compact, conservative, recursive method of integrating the equations of motion is proposed. All quantities in the equations of motion and in the field equations are either dimensionless or expressed in terms of units of length or inverse length.

Acknowledgement

This work was supported by the Swedish-Soviet Research Cooperation, established between the former USSR Academy of Sciences and the Royal Swedish Academy of Sciences.

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