Second Order Chromaticity of the Interaction Regions in the Collider

T. Sen and M.J. Syphers
Superconducting Super Collider Laboratory *
Dallas, TX 75237

Abstract

The collider in the SSC has large second order chromaticity (ξ_2) with the interaction regions (IRs) contributing substantially to it. We calculate the general expression for ξ_2 in a storage ring and find that it is driven by the first order chromatic beta wave. Specializing to the interaction regions, we show that ξ_2 is a minimum when the phase advance ($\Delta\mu_{IP-IP}$) between adjacent interaction points is an odd multiple of $\pi/2$ and both IRs are identical. In this case the first order chromatic beta wave is confined within the IRs. Conversely, ξ_2 is large either if $\Delta\mu_{IP-IP} = (2n+1)\pi/2$ and the two IRs are very far from equality or if the two IRs are equal but $\Delta\mu_{IP-IP} = n\pi$.

I. TUNE SHIFT AND CHROMATICITY TO 2ND ORDER

Consider a storage ring and label two points on it as 1 and 2. Let μ_0 be the global phase advance around the ring and (β_1, α_1, γ_1) the Twiss functions at point 1. The periodic transfer matrix at point 1 can be written as

$$M_1 = M(2\rightarrow 1)\cdot M(1\rightarrow 2) \qquad (1)$$

where $M(2\rightarrow 1)$ is the transfer matrix from point 2 to 1 etc. Let μ_1 and μ_2 be the phase advances at points 1 and 2 respectively with respect to an arbitrary reference point and $\mu_{21} = |\mu_2-\mu_1|$. We now introduce two infinitesimally thin quads of strengths $q_1 = k_1\Delta s_1$ and $q_2 = k_2\Delta s_2$ at points 1 and 2 respectively. Their perturbations to the transfer matrix are described by the matrices P_1 and P_2 where P_i is

$$P_i = \begin{bmatrix} 1 & 0 \\ -k_i\Delta s_i & 1 \end{bmatrix} \qquad (2)$$

These quad errors change the cyclic transfer matrix at point 1 to $\bar{M}_1$

$$\bar{M}_1 = M(2\rightarrow 1)\cdot P_2\cdot M(1\rightarrow 2)\cdot P_1 \equiv M_1 + \Delta M_1 \qquad (3)$$

Let $\Delta\mu$ be the change in the global phase advance around the ring. We scale the quad errors by an arbitrary parameter ϵ i.e. $k_1\rightarrow \epsilon k_1$, $k_2 \rightarrow \epsilon k_2$ and expand $\Delta\mu$ as a power series in ϵ,

$$\Delta\mu = \epsilon\Delta\mu_1 + \epsilon^2\Delta\mu_2 + \dots \qquad (4)$$

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The new global phase advance $\bar{\mu}_0 = \mu_0 + \Delta\mu$ is to be found from

$$\cos\bar{\mu}_0 = \frac{1}{2}\mathrm{Tr}\ \bar{M}_1 = \cos\mu_0 + \frac{1}{2}\mathrm{Tr}\ \Delta M_1 \quad . \qquad (5)$$

We also have

$$\cos\bar{\mu}_0 = \cos\mu_0\cos\Delta\mu - \sin\mu_0\sin\Delta\mu \quad .$$

Substituting Equation (4) into the above and equating it to the expression for $\cos\bar{\mu}_0$ given by Equation (5), we have

$$\frac{1}{2}\mathrm{Tr}\Delta M_1 = -\epsilon\sin\mu_0\Delta\mu_1 - \epsilon^2[\sin\mu_0\Delta\mu_2 + \frac{\cos\mu_0(\Delta\mu_1)^2}{2}] + O(\epsilon^3). \qquad (6)$$

To obtain the corrections to μ_0 order by order, we equate the coefficients of like powers of ϵ on both sides of the above equation. We can generalise to N quad errors in the ring and then take the limit of infinitesimally thin quads distributed around the ring of circumference C. In this limit, the 1st and 2nd order terms are,

$$\begin{aligned}\Delta\mu_1 &= \frac{1}{2}\int_0^C k(s)\beta_0(s)ds \\ \Delta\mu_2 &= \frac{1}{4\sin\mu_0}\int_0^C k(s)\beta_0(s)ds\int_s^C k(s')\beta_0(s') \\ &\quad\times[\cos\mu_0 - \cos(\mu_0 - 2|\mu(s')-\mu(s)|)]\ ds' \\ &\quad -\frac{1}{2}\cot\mu_0(\Delta\mu_1)^2 \end{aligned} \qquad (7)$$

Here we have let $\beta_0(s)$ denote the unperturbed β function at the point s. In the equation for $\Delta\mu_2$, we convert the integral over part of the ring to one over the complete ring and obtain

$$\begin{aligned}\Delta\mu_2 &= -\frac{1}{8\sin\mu_0}\int_0^C k(s)\beta_0(s)ds\int_s^{s+C} k(s')\beta_0(s') \\ &\quad\times\cos[\mu_0 - 2|\mu(s')-\mu(s)|]\ ds' \end{aligned} \qquad (8)$$

Recognizing that the integral over s' is related to the expression for the 1st order change in the β function [1], we obtain

$$\Delta\mu_2 = \frac{1}{4}\int_0^C k(s)\Delta\beta_1(s)ds \quad . \qquad (9)$$

This important relation tells us that the first order distortion in the β function propagating around the machine gives rise to the second order tune shift. The total phase shift to second order in the gradient errors is (after putting the arbitrary parameter ϵ to unity),

$$\Delta\mu = \frac{1}{2}\int_0^C k(s)\beta_0(s)ds + \frac{1}{4}\int_0^C k(s)\Delta\beta_1(s)ds + O(k^3) \quad . \tag{10}$$

The gradient perturbations of interest here are those seen only by particles off the design momentum. The chomatic error introduced by the quads is then corrected by placing sextupoles at places of non-zero dispersion. Assuming that only the horizontal dispersion D_x is non–zero, the effective quadrupole strengths in the horizontal and vertical planes for a particle with relative momentum deviation $\delta = \Delta p/p_0$ are respectively,

$$\begin{aligned} K_x^{eff} &= K_x(s,\delta) + S(s,\delta)D_x(s,\delta)\delta \\ K_y^{eff} &= K_y(s,\delta) - S(s,\delta)D_x(s,\delta)\delta \end{aligned} \tag{11}$$

As functions of δ, $K(s,\delta) = K_0(s)/(1+\delta)$ and $S(s,\delta) = S_0(s)/(1+\delta)$, where K_0 and S_0 are the nominal quad and sextupole strengths experienced by a particle on momentum. We expand D and β as power series in δ,

$$D(s,\delta) = D_0(s) + \Delta D_1^C(s)\delta + \Delta D_2^C(s)\delta^2 + \ldots \tag{12}$$

$$\beta(s,\delta) = \beta_0(s) + \Delta\beta_1^C(s)\delta + \Delta\beta_2^C(s)\delta^2 + \ldots \tag{13}$$

where the superscript C denotes a chromatic expansion. Hence the gradient error in the horizontal plane for the off-momentum particles is

$$k(s) = [S_0D_0 - K_0]\delta + [K_0 + S_0(\Delta D_1^C - D_0)]\delta^2 + O(\delta^3) \tag{14}$$

Substituting into Equation 10 and writing the tune shift in terms of the first and second order chromaticity ξ_1 and ξ_2 respectively,

$$\Delta\nu \equiv \frac{1}{2\pi}\Delta\mu = \xi_1\delta + \xi_2\delta^2 + O(\delta^3) \tag{15}$$

we obtain

$$\begin{aligned} \xi_1 &= \frac{1}{4\pi}\int_0^C \beta_0(s)[S_0(s)D_0(s) - K_0]\, ds \\ \xi_2 &= \frac{1}{8\pi}\int_0^C [S_0(s)D_0(s) - K_0]\Delta\beta_1^C(s)\, ds \\ &\quad + \frac{1}{4\pi}\int_0^C \beta_0(s)S_0(s)\Delta D_1^C(s)\, ds - \xi_1 \end{aligned} \tag{16}$$

The first order changes in β and D are given by

$$\begin{aligned} \frac{\Delta\beta_1^C(s)}{\beta_0(s)} &= \frac{-1}{2\sin\mu_0}\int_s^{s+C} [S_0(s')D_0(s') - K_0(s')]\beta_0(s') \\ &\quad \times \cos[\mu_0 - 2|\mu(s') - \mu(s)|]\, ds' \\ \Delta D_1^C(s) &= -\sqrt{\beta_0(s)}\int_s^{s+C} \frac{\sqrt{\beta_0(s')}}{\sin(\mu_0/2)}[S_0(s')D_0(s') - K_0(s')] \\ &\quad \times D_0(s')\cos[\frac{\mu_0}{2} - |\mu(s') - \mu(s)|]\, ds' \end{aligned} \tag{17}$$

Ignoring the phase factors for the moment, we see that $\Delta\beta_1^C$ which contains factors of $\beta(s)$ rather than $\sqrt{\beta(s)}$ (as occurs in ΔD_1^C) will dominate the contribution to the second order chromaticity. This situtation can change if we choose the phase advances between the major chromatic error sources appropriately. For example, two sources of equal strength $\pi/2$ apart in phase will produce β waves exactly out of phase so there will be no resultant β wave. The dispersion waves produced by the same two sources will add in quadrature. Alternatively, if we want to cancel the net dispersion wave, the two sources should be π apart in phase. In this case the β waves will add exactly in phase.

Hence to reduce the second order chromaticity, the first order changes in β and also in the dispersion D should be minimized. Conversely, the regions where $\Delta\beta_1$ is large (e.g. the triplets in the IRs) will contribute the most to the second order chromaticity. The above expression also exhibits the variation of ξ_2 with the global tune. Since the first order β wave diverges at integer and half–integer tunes, ξ_2 will be amplified as ν_0 approaches 0 or 0.5 and will be a minimum at ν_0=0.25 .

II. CHROMATICITY DUE TO IR TRIPLETS

The total chromaticity of an IR includes contributions from the triplets, the quads in the $M = -I$ section and the variable strength quads in the tuning section [2]. The triplets alone contribute 76% of this chromaticity at collision. Consequently we will consider the tune shift due to the chromatic error of the 4 IR triplets only and ignore the effect of other quadrupoles and sextupoles. Let

$$Q_i\beta_i \equiv \int_{ith\ triplet} K\beta\ ds \quad .$$

Then to 2nd order in the momentum deviation δ, the phase shift due to these 4 triplets is

$$\Delta\mu = \Delta\mu_1\delta + \Delta\mu_2\delta^2 + O(\delta^3) \tag{18}$$

where $\Delta\mu_1 = -1/2\sum_{i=1}^4 Q_i\beta_i$ and

$$\begin{aligned} \Delta\mu_2 &= \sum_{i=1}^{3}\sum_{j=i+1}^{4}\frac{Q_i\beta_iQ_j\beta_j}{4\sin\mu_0}[\cos\mu_0 - \cos(2\mu_{ji} - \mu_0)] \\ &\quad -\Delta\mu_1 - \frac{1}{2}\cot\mu_0(\Delta\mu_1)^2 \end{aligned} \tag{19}$$

μ_{ji} is the phase advance from the ith triplet to the jth triplet and $\nu_0 = \mu_0/2\pi$ is the global tune of the ring. The first order chromaticity is independent of phase advances between the triplets. However the second order chromaticity depends crucially on the relative phase advances between the triplets. If the phase advance between the IPs is $\Delta\mu_{IP-IP}$, then the relative phase advances have the following values,

$$\begin{aligned} &\mu_{21} = \pi, \quad \mu_{31} = \Delta\mu_{IP-IP}, \quad \mu_{41} = \Delta\mu_{IP-IP} + \pi \\ &\mu_{32} = \Delta\mu_{IP-IP} - \pi, \quad \mu_{42} = \Delta\mu_{IP-IP}, \quad \mu_{43} = \pi \end{aligned} \tag{20}$$

With these values, the second order contribution reduces to

$$\Delta\mu_2 = \Delta\mu_{2Q} + \frac{1}{2}(Q_1\beta_1 + Q_2\beta_2 + Q_3\beta_3 + Q_4\beta_4) \tag{21}$$

where $\Delta\mu_{2Q}$ is the contribution from terms second order in the quad strengths,

$$4\tan\mu_0\Delta\mu_{2Q} = (Q_1\beta_1 + Q_2\beta_2)(Q_3\beta_3 + Q_4\beta_4) \times\{1 - \frac{\cos(2\Delta\mu_{IP-IP} - \mu_0)}{\cos\mu_0}\} - \frac{1}{2}(Q_1\beta_1 + Q_2\beta_2 + Q_3\beta_3 + Q_4\beta_4)^2 \quad (22)$$

For arbitrary μ_0, the term in curly braces is a maximum and hence $\Delta\mu_{2Q}$ is a minimum if $2\Delta\mu_{IP-IP} = (2n+1)\pi$. Conversely $\Delta\mu_{2Q}$ is a maximum if $2\Delta\mu_{IP-IP} = 2n\pi$. The large β functions in the triplets ensures that $\Delta\mu_{2Q}$ completely dominates the contribution to $\Delta\mu_2$. Hence choosing $\Delta\mu_{IP-IP} = (2n+1)\pi/2$ minimizes the 2nd order chromaticity of the IRs. This is due to the fact that the chromatic β waves from the IRs are exactly out of phase and interfere destructively. The following discussion will assume this choice of $\Delta\mu_{IP-IP}$.

An exact cancellation of the β waves occurs if the two IRs have the same β_{peak}. In this configuration, the repetitive symmetry across the two IRs implies $Q_3\beta_3 = Q_1\beta_1$, $Q_4\beta_4 = Q_2\beta_2$. $\Delta\mu_{2Q}$ vanishes as a consequence of the fact that the β wave is zero outside the triplets. The entire 2nd order phase shift is

$$\Delta\mu_2 = (Q_1\beta_1 + Q_2\beta_2) \quad . \quad (23)$$

For this case alone, $\Delta\mu_2$ is independent of the global tune ν_0.

In the following table, we evaluate the 2nd order chromaticity due to the triplets in three different configurations and at two tunes.

Table 1 : 2nd order chromaticity due to the triplets

Case	ξ_2	
	$\nu_0 = 0.285$	$\nu_0 = 0.4$
I) Equal IPs		
β^*=0.25m	154.0	154.0
β^*=0.50m	77.0	77.0
II) Unequal IPs		
β^*=0.25m, β^*=0.50m	1156.4	6524.8
III) One IP		
β^*=0.25m, β^*=8.00m	3977.5	24132.6

For all cases except the first, the second order chromaticity is a minimum at $\nu_0 = 0.25$ and will be significantly amplified as $\nu_0 \rightarrow 0.5$.

The chromaticity correction scheme proposed for the IRs is discussed in [3]. Briefly, sextupoles are placed in 24 arc cells adjacent to the cluster containing the two IRs. The third case in Table 1 at tunes (123.435,122.416) has large ξ_2 and requires nonlinear correction. For this configuration, Figure 1 shows the chromatic β wave (at δ=0.0003) through the cluster and adjacent cells without the nonlinear correction. Figure 2 is the corresponding figure after the nonlinear chromaticity is corrected. The β beat in the arcs is reduced from 75% in Figure 1 to 2% in Figure 2. This clearly illustrates the connection between the chromatic β wave and the nonlinear chromaticity.

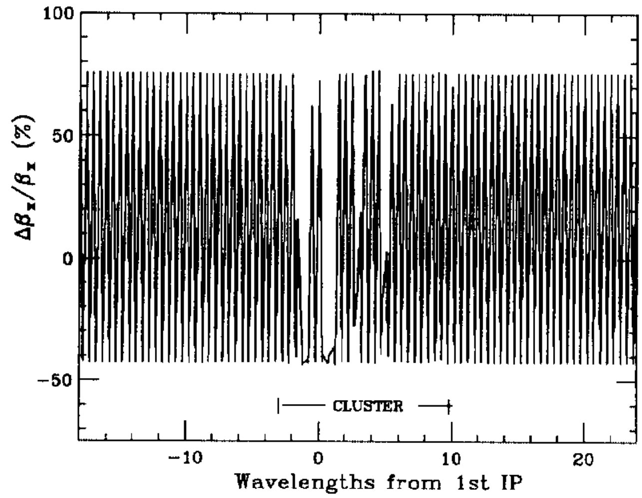

Figure 1. Horizontal Chromatic β beat without nonlinear chromaticity correction

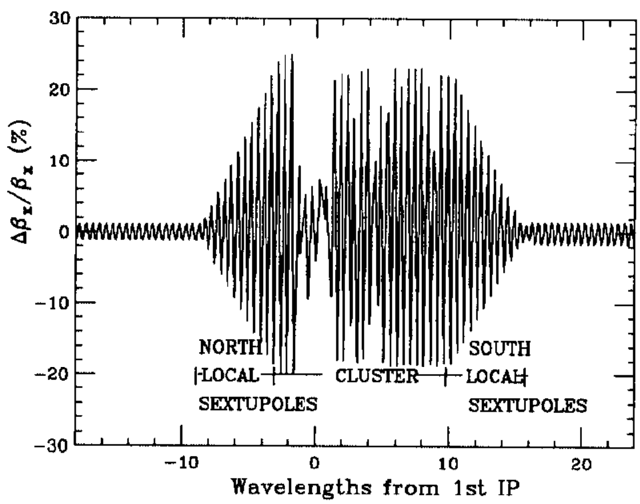

Figure 2. Horizontal Chromatic β beat with nonlinear chromaticity correction

III. REFERENCES

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