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FIELD PROCESSING METHODS DEVELOPED FOR A SEPARATED-SECTOR CYCLOTRON

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Summary

This paper describes three numerical techniques which were developed during beam orbit studies undertaken for the separated-sector cyclotron of the NAC project[1]. The techniques are concerned with the numerical manipulation of field map data. Firstly, a method is given for interpolating the field and calculating its derivatives more accurately than is possible by conventional methods, in regions where the field varies rapidly and the data are not closely spaced. The technique consists of fitting a suitable analytical function f(x) to the field distribution $B_i = B(x_i)$ so as to minimize the fourth differences of the distribution $B_i - f(x_i)$. The distribution $B_i - f(x_i)$ is then smooth enough for conventional methods to be applied. Secondly, we describe a method for isochronizing field map data, the method being applicable to magnets which have trim-coils with edge contours similar to the shapes of the equilibrium orbits within the sectors. Thirdly, we discuss a modification of an alternative isochronization technique[2,3], this method being applicable to magnets with trim-coils having edges which are arcs of circles concentric about the centre of the cyclotron.

Introduction

During the early phases of the NAC project it was decided that for design study purposes realistic numerical field maps of the cyclotron magnets would be obtained by means of accurate field calculations[4]. In the case of the sector magnets of the separated-sector cyclotron (SSC) the map data are values of the field in the median plane at the nodes of a cylindrical grid. Both the memory requirements and the processing time needed to run the field calculation programs placed limits on the number of node points that could be used, and the first sector magnet fields were calculated with radial and azimuthal spacings Δr and $\Delta\theta$ of 73 mm and 1° respectively. On the other hand, a sector magnet pole gap of 60 mm had been chosen on the basis of focusing requirements. Since both Δr and $r\Delta\theta$ (at large radii) are larger than the pole gap, the magnetic field at the magnet edge is represented by relatively few map data. A typical radial distribution in the vicinity of the magnet edges is shown in figure 1. The azimuthal distributions at large radii display a similar rapid variation of the field at the magnet edge. When conventional techniques for interpolating or calculating derivatives are applied to such distributions, they fail to give accurate results:- cubic splines fitted to the radial distributions tend to oscillate strongly in the vicinity of the field edges. Similarly a Fourier decomposition of the azimuthal distributions followed by a reconstruction of a smoothed field from the Fourier expansion formula in which Lanczos smoothing coefficients are included, has the effect of broadening the field edge and decreasing $|\partial B/\partial\theta|$.

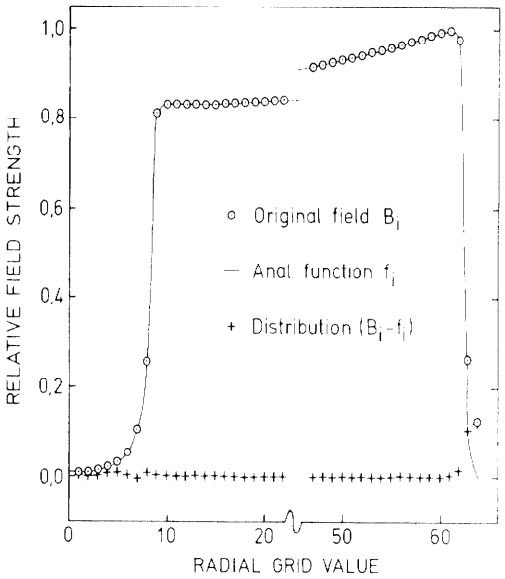

Fig. 1: Typical radial distribution in the vicinity of the magnet edges.

A Method of Interpolating and Calculating the Derivatives of a Rapidly Varying Field

The reason why conventional techniques fail when used on the abovementioned type of distribution becomes clear when the n-th central differences $\delta^n B_i$ of the distribution B_i are calculated: the $\delta^n B_i$-values do not become small but increase as n is increased. If one wishes to represent N equally spaced values of B_i of a smooth function by an (n-1)th order polynomial (N>n) then the n-th differences $\delta^n B_i$ should become (vanishingly) small everywhere along the distribution. Therefore, for the given spacing, the field edges cannot be represented by a polynomial.

The basis of the method reported here is that a suitable function f(x) (where x represents r or θ) can be fitted to the distribution B_i in such a way that the distribution of the difference $B_i - f(x_i)$ is smooth. The latter distribution will be smooth if the higher order differences $\delta^n(B_i - f_i)$ are (vanishingly) small. In particular, if a collocating cubic spline s is to be fitted to the distribution $B_i - f_i$, and if the spline function is to be non-oscillatory, then the quantities

$$d^3 s_i/dx^3\big|_{x_i + 0} - d^3 s_{i-1}/dx^3\big|_{x_i - 0}$$

(where s_i and s_{i-1} are the piecewise cubic polynomials to right and left of the nodes x_i) should be small for all x_i. This condition is met if the fourth differences $\delta^4(B_i - f_i)$ are small everywhere. Therefore we fit the function f(x) to the distribution B_i so as to minimize the sum of the squares of the fourth differences $\delta^4(B_i - f_i)$. The function f(x) will reduce $\delta^4(B_i - f_i)$ at the field edge only if its shape is a good approximation of the field edge shape. The formfactor we have chosen to describe the field edge j is:

$$g_j(x) = \left[1 + \{\alpha_j + (2/\pi)\beta_j \tan^{-1}\gamma(x - \bar{x}_j)\}\,(x - \bar{x}_j)\right]^{-1} \quad (1)$$

where α_j and $\bar{x}_j$ are measures of the width and position of the field edge, and β_j and γ are introduced to provide some asymmetry. The plateau regions j of the distribution are described by polynomials

$$P_j(x) = \sum_{k=0}^{n_j} a_{jk} x^k.$$

The function f(x) is then constructed in a modular fashion from the functions P_j and g_j, e.g., for the radial distribution with two field edges shown in fig. 2:

$$f(x) = P_o g_1 + P_1(g_2 - g_1) \quad (2)$$

In practice only those parameters of f(x) which materially affect the $\delta^4(B_i - f_i)$ values are varied; they are α_j, β_j, and $\bar{x}_j$ of the formfactors g_j. The coefficients of the polynomials P_j are calculated directly from the data of the plateau regions and are then clamped, whereas γ is found by trial and error. When the $\delta^4(B_i - f_i)$-values have been minimized, we fit a cubic spline s (for radial as well as azimuthal distributions) to the distribution $B_i - f_i$, by means of a least-squares method due to Dierckx[5]. An interpolation of B at point x is then given by B(x) = s(x) + f(x) and the derivative of B at x by $\delta B/\delta x = ds/dx + df/dx$.

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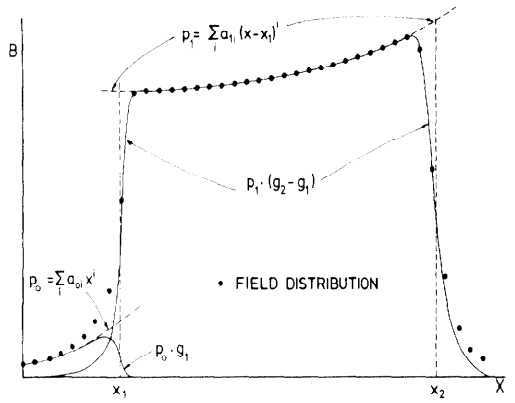

Fig. 2: Illustration of the manner in which the function f(x) is chosen for a radial distribution

The technique gives results which are superior to those of conventional methods when the spacing of the data as described above pertains. When the data are spaced sufficiently closely, conventional methods and the present one agree very closely. Figure 1 shows a typical fit of the function f(x) to radial data and the resulting smoothness of the $(B_i - f_i)$ distribution. Figure 3 shows the derivatives $\delta B/\delta\theta$ obtained by the Fourier analysis method (circles) and the present method (solid line) for a typical azimuthal distribution.

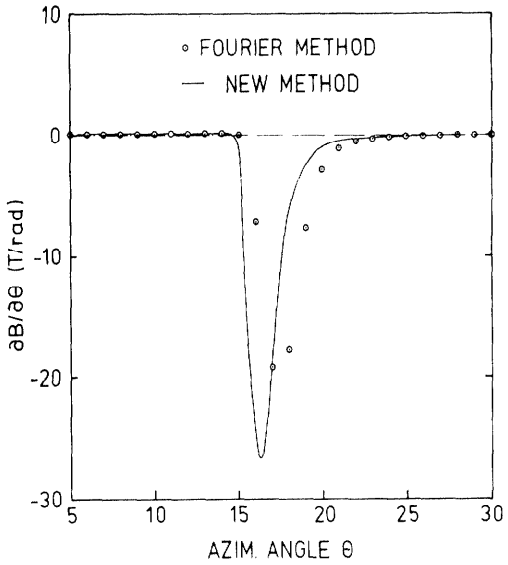

Fig. 3: Derivatives of an azimuthal distribution at the magnet edge.

A Simple Method for Isochronizing Magnetic Field Maps

Most published methods dealing with the isochronization of numerical field maps[2,3] assume that the field is varied as a function of radius only, and that the azimuthal field profiles remain constant. These methods apply to magnets in which the inner and outer edges of the trim-coils are arcs of circles concentric about the centre of the machine. The trim-coils of the SSC sector magnets, however, are contoured to the same shape as the equilibrium orbits (e.o.) within the sectors, i.e., their radii of curvature agree with those of the e.o.'s within the sectors. It follows that both the radial and azimuthal field profiles will change if the trim-coil currents are varied. We have developed an isochronization method which will approximately take the azimuthal variation into account: essentially the same method has been developed independently at GANIL[6]. The basic assumption of the method is that the field on the e.o. is varied by a constant fraction of its local value. Thus if B(s) is the field value at a distance s along the e.o., a variation of the trim-coil current causes a variation dB(s) of the field such that $dB(s)/B(s) = db$ is a constant for all s along the e.o. The change $d\tau$ in the orbital period τ is then related to the fractional change db by the equation

$$d\tau/\tau = -(1/\gamma^2)db, \qquad (3)$$

γ being the relativistic ratio of total to rest energy. Equation (3) is an exact result. The advantage of varying the field in this manner is that the shape of an e.o. at a particular radius does not change during isochronization. Equation (3) is now assumed to hold for finite changes $\Delta B(s)/B(s) = b$ of the field on the e.o. For the i-th e.o. we can write, therefore:

$$b_i = -\gamma_i^2 \Delta\tau_i/\tau_i = -\gamma_i^2(\tau_o - \tau_i)/\tau_i \qquad (4)$$

where τ_i is the orbital period of the i-th e.o., and b_i the fraction with which, to first order, the field on the e.o. must be increased to isochronize it to the period τ. However, b_i applies to the field on the i-th e.o. and not at the nodes of the grid. To calculate the fractional changes b applicable at the nodes we proceed as follows:

a) We calculate the b_i-values of m e.o.'s, evenly distributed over the radial range within which isochronism is desired, m normally being larger than the number of radial nodes enclosed within the range.

b) At every nodal value of the azimuthal angle the b_i are treated as dependent variables of the radius values r_i of the e.o.'s at that angle (see fig. 4). A cubic spline is fitted to b_i in a least-squares sense, according to the prescription of Dierckx[5], so as to average out errors in b_i introduced by the numerical calculations of τ_i.

c) For radial node values R_j which fall between the radius values r_1 and r_m of the inner- and outermost orbits, the b_j-values at the nodes are taken to be the values of the spline function s, viz., $b_j = s(R_j)$. For node radii $R_j < r_1$ or $R_j > r_m$ the b_j are calculated by a reasonable extrapolation of the spline function: the possibilities incorporated in our computer program are shown in figure 4.

We find that the method gives excellent results, provided that the map data are reasonably smooth (i.e., do not oscillate). Usually it is possible to reduce $\Delta\tau/\tau$ within two or three iterations to 2×10^{-5} or less, while the cumulative deviation from isochronism anywhere in the isochronized region is of the same order of magnitude.

Modified Alternative Method of Isochronizing Field Maps

The original isochronization program written within our group was based on the method in use at SIN[2]. This method assumes that the field is varied as a function of radius only while the azimuthal profiles remain unchanged. The radial change within which isochronism is desired is divided into a number of intervals n and the average field within the intervals is varied. The change $d\tau_i$ in the orbital period τ_i of the i-th e.o. is then expressible in terms of the variation $\Delta\bar{B}_j$ of the average field $\bar{B}_j$ by the matrix equation

$$(\Delta\tau/\tau)_i = -(1/\gamma_i^2)\sum_{j=1}^{n} A_{ij}(\Delta\bar{B}_j/\bar{B}_j) \qquad (5)$$

where the matrix elements A_{ij} are found by numerical integration. If in addition there exist n values of $(\Delta\tau/\tau)$, the matrix equation (5) can be inverted to yield values of $\Delta\bar{B}/\bar{B}$ which to first order should isochronize the field. In common with others[2,3] we have found that the inversion of equations (5) very often gives very unstable results. The instability of the solutions can be traced to two factors:

a) unless the relative positions of the e.o.'s and the radial intervals are chosen very carefully, the

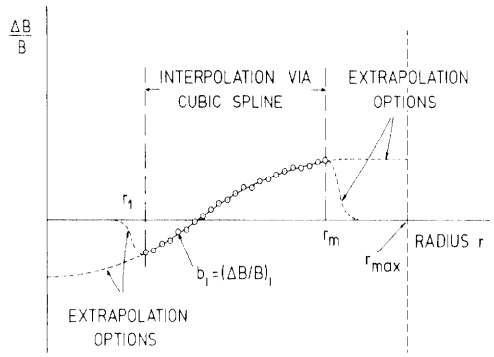

Fig. 4: Illustration of the fit of a least-squares cubic spline to the calculated b_i-values.

determinant of A is very nearly zero, i.e., the matrix is ill-conditioned for inversion. The resulting solutions $\Delta\overline{B}/\overline{B}$ are then very sensitive to small changes in $\Delta\tau/\tau$;

b) both the matrix elements A_{ij} and the orbital periods τ_i contain errors arising from the approximations inherent in numerical integration.

We have modified the method so that the problems associated with it are largely eliminated. As before, the radial range within which isochronism is desired is divided into a number of intervals n, except that the interval j is represented by formfactor $g_j(r)$ which we choose to be the cubic B-spline function[7] (see fig. 5). The knots of the B-splines are usually chosen to coincide with the nodes of the radial grid. If the magnetic field is written in the form $B(r,\theta) = B_o(r)\cdot f(r,\theta)$, where $B_o(r)$ is the radial distribution at $\theta = \theta_o$ (e.g., on the hill) and $f(r,\theta)$ describes the azimuthal behaviour of the field, then the radial distribution $B_o(r)$ is varied according to the prescription

$$B'_o(r) = B_o(r)\left[1 + \sum_{j=1}^{n} b_j\, g_j(r)\right] \qquad (6)$$

where the coefficients b_j are the relative amplitudes of the formfactors g_j. The formfactors ensure that $B_o(r)$ varies smoothly with r in any isochronization process. For any position s along an e.o. the change $\Delta B(s)$ in the field is thus given by

$$\Delta B(s) = \left[B'_o(r) - B_o(r)\right] f(s) = B(s) \sum_{j=1}^{n} b_j\, g_j(r) \qquad (7)$$

and the expression for the fractional change $(\Delta\tau/\tau)_i$ of the i-th e.o. becomes

$$\left(\frac{\Delta\tau}{\tau}\right)_i \simeq -\frac{1}{\gamma_i^2}\,\frac{\sum_{j=1}^{n}\left[\int_0^{L_i} B(s_i)\, g_j(r_i)\, ds_i\right] b_j}{\left[\int_0^{L_i} B(s_i)\, ds_i\right]} \qquad (8)$$

$$= \sum_{j=1}^{n} A_{ij}\, b_j$$

where the subscripts i and j refer to quantities associated with orbit i and interval j respectively, s_i is the distance along orbit i and L_i the total length of orbit i. The integrals of equation (8) are found by numerical integration.

So as to average out non-systematic errors occurring in A_{ij} and $(\Delta\tau/\tau)_i$, many more e.o.'s m are calulated than there are intervals n. The set of equations (8) is therefore overdetermined with respect to b_j. Setting $(\Delta\tau/\tau)_i = (\tau_o - \tau_i)/\tau_i = t_i$, we calculate b_j from the condition that $B_o(r)$ is varied in such a way as to minimize

$Q = \sum_{i=1}^{m} t_i^2$ The least-squares condition is written as

$Q_o = \sum_{i=1}^{m} t_{io}^2$ = minimum, where the subscript o

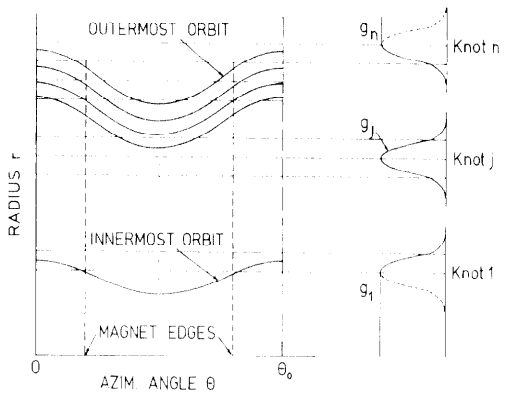

Fig. 5: Illustration of how the intervals j and formfactors g_j are chosen for the modified isochronization method.

indicates the values of t_i which minimizes Q. It is easy to show by differentiating Q_o with respect to b_j, that the least-squares condition leads to the following set of approximate equations:

$$\sum_{k=1}^{n}\left(\sum_{i=1}^{m} A_{ij}\, A_{ik}\right) b_k = -\sum_{i=1}^{m} t_i\, A_{ij} \qquad (9)$$

or $\sum_{k=1}^{n} M_{jk}\, b_k = N_j \qquad j = 1,2.....n$

which can be inverted to yield values of b_k. If one notes that the diagonal elements $M_{jj} = \sum_{i=1}^{m} A_{ij}^2$ refer to the contributions of all e.o.'s to a single interval j, whereas the off-diagonal elements

$$M_{jk} = \sum_{i=1}^{m} A_{ij}\, A_{ik}$$

refer to contributions to two at most partially overlapping intervals j and k, one can deduce that the spacing of the intervals must not be much smaller than the radial interval occupied by an e.o. in the sector, if the diagonal elements are to dominate. Also the inner- and outermost intervals must straddle those regions of r of the inner-, and outermost e.o.'s l and m for which the magnetic field on the orbits ($B(s_l)$ and $B(s_m)$) will contribute significantly to the upper integral of equation (8). Under these conditions the determinant of M is not small, and the inversion of equation (9) yields stable and well-behaved values of b_k.

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