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SYNCHROBETATRON OSCILLATION DRIVING MECHANISM*

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SUMMARY

Any cavity-like structure which supports modes that have both a finite accelerating field at the location of a particle beam passing through the structure and a transverse gradient in that accelerating field is capable of driving synchrobetatron oscillations. Such oscillations are incoherent, are substantially independent of the chromaticity, and do not depend explicitly on the η-function. Methods of computing the amplitudes of such oscillations, which can be quite significant under some circumstances, are presented.

INTRODUCTION

Either an RF cavity or an incidental vacuum chamber cavity in an electron-positron storage ring is capable of causing incoherent synchrobetatron oscillations if it has deflecting modes whose axes do not coincide with the actual beam axis. Modes having transverse first-order gradients in their longitudinal accelerating fields are well known to have magnetic fields which cause a net deflection of the beam, and are hence called deflecting modes. Separation of the actual beam axis from the mode axis can be caused either by having the beam off center in the vacuum chamber, or by having an asymmetric cavity. By asymmetric cavity is meant a cavity whose boundary, where it intersects a plane perpendicular to the nominal beam axis, is not equidistant in the $\vec{r}$ and $-\vec{r}$ directions (the symbols used in this paper are defined in Appendix I.)

The physical mechanism which causes the synchrobetatron oscillations is as follows. A charged bunch passing through a cavity off-axis relative to a deflecting mode will induce a voltage in that mode. Associated with the rate of change of this voltage is a deflecting magnetic field which will cause angular deflections of particles in the bunch. The magnitude of these deflections depends on the longitudinal position within the bunch of each particle. On successive passages through the cavity, the location of each particle relative to the bunch center will be governed by that particle's normal synchrotron oscillations, which are periodic with frequency f_s. Nonlinearities in the synchrotron motion (due to a sinusoidal, rather than linear, restoring force; potential well distortion in the RF cavities; and higher modes in various structures which produce longitudinal forces which are not linear with z) and in the deflection (because the deflecting magnetic field varies sinusoidally, or in an even more complicated way, rather than linearly, with z) cause deflections at various harmonics of f_s. The magnitudes of these harmonics depend on the amplitude of a given electron's synchrotron oscillation. Since various electrons with the same synchrotron oscillation amplitude have random synchrotron oscillation phases, the resulting deflections are incoherent. When

$$f_\beta \pm m f_s = M f_0 \qquad (1)$$

the necessary condition is met for the betatron oscillation amplitude to grow linearly with time, to be limited only by Landau damping, radiation damping, or by physical boundaries of the beam chamber. Since Landau damping arises largely from variations in synchrotron oscillation amplitudes, and attendant variations in f_s and (through the chromaticity) f_β, and since the present mechanism deals with particles of particular synchrotron oscillation amplitudes, Landau damping is not expected to be significant in this case, unless the lattice has an appreciable octupole moment. In the absence of Landau damping and beam loss, the linear growth rate will be balanced by the radiation damping rate.

HIGH-Q CASE

Consider a deflecting mode with a sufficiently high Q that the majority of the stored energy is retained between bunch passages. For simplicity, assume that the synchrotron oscillations and betatron oscillations are sinusoidal. Also assume that the condition

$$f = k \cdot f_0 \qquad (2)$$

is met. Denote the separation between the mode axis and the actual beam axis by x. Under these assumptions, an electron which is executing synchrotron oscillations of amplitude A will appear in the cavity with longitudinal excursion

$$z = A \cos(f_s \cdot 2\pi n/f_0) \qquad (3)$$

To first order, a deflecting mode has a transverse gradient in the longitudinal effective voltage, which will be written as V'. This corresponds to a quadratically rising effective shunt impedance, R''. For a gaussian bunch of N particles,

$$V' = NeR'' \, x \, f_0 S \qquad (4)$$

where S is the reduction factor $\exp(-4\pi^2 f^2 \sigma^2/c^2)$ resulting from the spread of the bunch.

Now consider a path integral along the mode axis, returning a distance x off axis. It follows directly from Maxwell's equations that

$$\left|\int B_y \cdot x \, dz\right| = \left|V' x / (2\pi f)\right| \qquad (5)$$

where B_y includes the same transit time factors as V'. The angular deflection of a particle is given by

$$\Delta\theta_{max} = \left|\frac{e}{m_0 \gamma c}\int B_y dz\right| = \left|\frac{e V'}{2\pi m_0 \gamma c f}\right| \qquad (6)$$

for an electron with suitable phase.

Since the center of the bunch is maximally retarded in the high-Q limit, with eq. (2) satisfied, the phase relationship between $\vec{\varepsilon}$ and $\vec{B}$ dictates that an electron which arrives slightly before the bunch center will be deflected toward the $\varepsilon_z = 0$ line, an electron at the bunch center will not be deflected, and an electron which arrives slightly after the bunch center will be deflected away from the $\varepsilon_z = 0$ line.

The phase Φ (relative to the mode) with which an electron arrives is $\Phi = 2\pi f z/c$. The deflection it receives is

$$\Delta\theta = -|\Delta\theta|_{max} \cdot \sin\phi = -\frac{e V'}{2\pi m_0 \gamma c f} \sin\left(\frac{A\, 2\pi f}{c} \cdot \cos\left(2\pi n \frac{f_s}{f_0}\right)\right) \qquad (7)$$

In order to determine the dependence on various harmonics of f_s, define $\rho = 2\pi f/c$ and $\phi = 2\pi n f_s/f_0$, and expand the sine in a power series. Next reduce the powers of cosines to multiple angles. This yields, keeping up to fifth power in ρ,

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$$\sin(\rho \cos\phi) \approx (\rho - \frac{\rho^3}{8} + \frac{\rho^5}{192})\cos\phi + (\frac{-\rho^3}{24} + \frac{\rho^5}{384}) \cdot$$

$$\cos(3\phi) + (\frac{\rho^5}{1920})\cos(5\phi) + \ldots \qquad (8)$$

Combining equations (4) and (7),

$$\Delta\theta = -\frac{Ne^2 x f_0 R''S}{2\pi m_0 \gamma c f} \sin(\rho \cos\phi) \qquad (9)$$

If the electron under consideration is oscillating with a betatron angle $\theta = -\theta_0 \cos(2\pi f_\beta t)$, and equation (1) is satisfied, θ_0 will grow in amplitude at the rate

$$\frac{d\theta_0}{dt} = \frac{N e^2 x f_0^2 R'' S}{4\pi m_0 \gamma c f} F_m(\rho)$$

where the function $F_m(\rho)$ denotes the appropriate harmonic coefficient, and the extra factor of 2 in the denominator results from averaging over a $\cos^2$ term in adding $\Delta\theta$'s to θ_0. The maximum betatron excursion (in the sinusoidal betatron motion approximation) is related to θ_0 by $X_0 = \theta_0 c/(2\pi f_\beta)$. Hence

$$\frac{dX_0}{dt} = \frac{N e^2 x f_0^2 R''S F_m(\rho)}{8\pi^2 m_0 \gamma f_\beta f} \qquad (10)$$

This linear growth rate competes with the damping $X_0 = X_{00} e^{-\delta t}$. The equilibrium X_0 is thus

$$X_0 = \frac{N e^2 x f_0^2 Z''S F_m(\rho)}{\delta 8\pi^2 m_0 \gamma f_\beta f} \qquad (11)$$

Equation 11 can be generalized by including the nonlinearity of the synchrotron oscillations in the expressions for $F_m(\rho)$, (which, in general, will introduce even harmonics and increase the magnitudes of the higher harmonic coefficients) and by using the "instantaneous" betatron frequency at the location of the cavity for f_β; for example, if one is considering vertical betatron oscillations and is concerned with a cavity in a region in which the β-function is K times its average value, the instantaneous value of f_β would be $\sqrt{K}$ times smaller than the average value, leading to an X_0 (in the normal part of the lattice) $\sqrt{K}$ times as large as would otherwise have occurred.

LOW-Q CASE

As shown in Appendix II, the fields just following passage of the bunch are, for a single mode, higher in the high-Q case than in the low-Q case by the ratio $Q f_0/(\pi f)$. Hence for the low Q case,

$$X_0 = \frac{N e^2 x f_0 Z''S F_m(\rho)}{\delta 8 \pi m_0 \gamma f_\beta Q} \qquad (12)$$

In the low-Q case, the mode is unexcited prior to arrival of the bunch, and maximally excited (in general) just after the bunch leaves (note that this same contribution is present in the high-Q case, but that the residual sinusoidal fields from previous bunch passages are superimposed on the low-Q fields, and are dominant for sufficiently high Q). Substantial modifications have to be made to the function $F_m(\rho)$ to account for the rapidly changing mode excitation in the low-Q case. In addition, all deflecting modes of the cavity will contribute to the synchrobetatron oscillation, since equation (2) no longer has to be satisfied.

QUANTITATIVE EXAMPLES

A high-Q situation is considered in which the equivalent of a CESR RF cell has its axis shifted 0.04 m relative to the beam axis; the equivalent of this situation could occur if a large, asymmetric vacuum box were present in the ring. For an actual RF cell, x should be less than 0.01 meters if good position monitors are available and reasonable care is taken in aligning the beam. For a CESR cell, the author has measured a TM_{111} deflecting mode shunt impedance, appropriately reduced for a gaussian bunch with $\sigma = .045$ m, of $Z''S = 3.371 \cdot 10^8 \Omega/m^2$. (Some authors prefer to normalize such a shunt impedance to λ or $\bar{\lambda}$, thus eliminating the m^{-2} dependence). The frequency of the TM_{111} mode is $f = 1.138\ 10^9$Hz, and the Q is $4 \cdot 10^4$. Consider those electrons whose rms synchrotron oscillation excursion is 2σ. Although relatively few electrons have this large an excursion, if those electrons which have this excursion are lost, the beam lifetime will be extremely short (of the order of 200 radiation damping time constants). Other numbers relevant to three examples are given in Table I, together with the values of X_0 which result for various harmonics $F_m(\rho)$.

Machine	Case I CESR	Case II CESR	Case III SPEAR II	
Energy	8	3	1.5	GeV
N	$1.5\ 10^{12}$	$8.2\ 10^{11}$	$1.95\ 10^{11}$	
f_0	$3.9\ 10^5$	$3.9\ 10^5$	$1.28\ 10^6$	Hz
δ	250	13.2	6.37	sec^{-1}
f_β	$4.438\ 10^6$	$4.438\ 10^6$	$6.66\ 10^6$	sec^{-1}
X_0(m=1)	.049	1.35	9.56	m
X_0(m=3)	.027	.76	5.35	m
X_0(m=5)	.0074	.205	1.45	m

The three examples given in Table I assume that the cavity is in a region of average β. As previously discussed, the values of X_0 would be five times as high as those shown if the cavity were in a location where β had 25 times its average value, as might be found for β_v near a low-β insertion.

The probability that a single cell of the type considered will have a particular mode which, within its bandwidth, satisfies equation (2) is 0.073 for CESR and .24 for SPEAR II. If the machine has several such cells, each with several important deflecting modes, the probability that one of them will satisfy equation (2) becomes quite high. In the low-Q cavity case, the values of X_0 would, for low harmonic numbers, be approximately four times smaller than those shown for CESR, and fourteen times smaller for SPEAR II. Factors which would tend to increase the values of X_0 are the contribution of all deflecting modes (not just the one with frequency f)and, for higher harmonics, the rapidly changing energy in the cavity during the bunch passage (which would increase the amplitudes of higher harmonic coefficients).

CONCLUSION

Incoherent synchrobetatron oscillations, driven by deflecting modes in asymmetric cavities or in symmetric cavities through which the beam passes off-center, can lead to severe beam loss under some conditions, as seen in the examples of Table 1. Both high-Q and low-Q contributions are important. It appears that the best methods for avoiding this problem are to avoid unnecessary cavity-like structures, to avoid satisfying equation (1), to keep ν as far as possible from an integer to make the values $F_m(\rho)$ small, to make necessary cavity-like structures as symmetrical as possible, and to center the beam as well as possible, particularly in high-β regions.

The measurable properties of synchrobetatron oscillations due to this mechanism are difficult to distinguish from those due to the mechanism described by the SPEAR group,[1] and both may be important.

Appendix I

Symbols

Standard International Units Are Used

- A Amplitude of synchrotron oscillation, meters
- B_y Magnetic field normal to the plane of a betatron oscillation.
- c Speed of light
- e Charge of an electron
- f Frequency of a cavity mode
- $F_m(\rho)$ Coefficient for the mth harmonic of f_s
- f_0 Bunch revolution frequency
- f_s Synchrotron oscillation frequency
- f_β Betatron oscillation frequency
- K Ratio of local β to average β
- k Integer
- L Length of a cavity cell
- M Integer
- m Integer
- m_0 Mass of an electron
- N Number of particles in bunch
- n Number of bunch revolutions completed
- P Power dissipated in a cavity cell
- Q Ratio of stored energy to energy dissipated per radian
- q Charge in a very short bunch
- R Shunt impedance of a cell
- $\vec{r}$ In cylindrical coordinates, the radius vector
- R'' Shunt impedance is $R'' x^2$
- S Gaussian bunch shunt impedance reduction factor
- T Cell transit time factor
- TM_{111} Transverse magnetic mode with two azimuthal zeroes, one radial zero at wall, and one longitudinal zero in ε_z.
- t Time
- U Stored energy
- V' Effective voltage = $V'x$
- x Distance transverse to beam axis
- X_0 Amplitude of betatron oscillation
- X_{00} Initial amplitude of betatron oscillation
- Z Shunt impedance for infinite velocity particles
- z Distance in the beam direction from the bunch center
- β Function which prescribes betatron trajectories
- β_V β in the vertical plane
- γ Ratio of total mass to rest mass
- $\Delta\theta$ Change in betatron angle
- δ Radiation damping rate, betatron motion
- η Function which describes off-energy closed orbit
- θ Betatron angle
- θ_0 Maximum betatron angle
- λ Wavelength
- $\lambda\!\!\!^-$ $\lambda/(2\pi)$
- ν Average number of betatron oscillations per revolution
- ε Electric field
- ε_f Final ε after bunch passes
- ε_i Initial ε before bunch passes
- ε_z Component of ε in the z direction
- ρ $2\pi f/c$
- σ Standard deviation in meters of gaussian bunch
- τ Bunch revolution time
- Φ Phase of a particle relative to phase of a deflecting mode
- ϕ $2\pi n f_s/f_0$
- ω $2\pi f$

Appendix II

Relationship Between High-Q and Low-Q Fields

In a cavity, the relationships between power, field, shunt impedance, and stored energy are

$$\frac{P}{L} = \frac{\varepsilon^2}{R/L} = \frac{\varepsilon^2}{(ZT^2/Q)Q} = \frac{\varepsilon^2}{(ZT^2/Q)U/(P/\omega)} \qquad (13)$$

$$U = \frac{\varepsilon^2}{(ZT^2/Q)(P/L)/(P/\omega)} = \frac{\varepsilon^2}{(ZT^2/Q)(\omega/L)} = \frac{\varepsilon^2 L}{(ZT^2/Q)\omega} \qquad (14)$$

Taking the derivative of U with respect to q, and using conservation of energy,

$$\frac{dU}{dq} = \varepsilon L = \frac{2\varepsilon(d\varepsilon/dq)L}{(ZT^2/Q)\omega} \qquad (15)$$

Eliminating εL from the right hand equation in (15) and integrating yields

$$\int d\varepsilon = \int \frac{ZT^2\omega}{2Q}\,dq \qquad (16)$$

Due to the combination of fields in the cavity and fields added in phase by the bunch passage, there results

$$\varepsilon_f = \varepsilon_i + \frac{ZT^2}{Q}\,\frac{\omega}{2}\,q \qquad (17)$$

Due to field decay between bunch passages, there results

$$\varepsilon_i = \varepsilon_f \exp(-\tau\omega/2Q) \qquad (18)$$

Combining (17) and (18) yields

$$\varepsilon_f = (ZT^2/Q)(\omega/2)q/\ (1 - \exp(-\omega\tau/(2Q)) \qquad (19)$$

For very high Q, $\exp(-\omega\tau/(2Q)) \simeq 1 - \omega\tau/(2Q)$, so (19) becomes

$$\varepsilon_f = (ZT^2/Q)Q(q/\tau) \qquad (20)$$

(Note that this is another way of writing Ohm's law.) (20)

Dividing (20) by (17), with ε_i set equal to 0 to represent the low Q case, the high Q to low Q enhancement factor $2Q/(\omega\tau)$ results.

REFERENCE

1. SPEAR group, "SPEAR Performance," Proc. 1975 Particle Accelerator Conference, Washington, D.C., March 12-14, 1975 (IEEE Trans. Nucl. Sci. NS-22, No. 3, 1366 (June 1975)).