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BEAM EMITTANCE GROWTH IN A PROTON STORAGE RING EMPLOYING CHARGE EXCHANGE INJECTION*

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Introduction

Recently it has been shown that very large currents can be accumulated in medium energy proton storage rings by multiturn injection of an H⁻ beam through a charge stripping medium.[1] Since the particles are injected continuously into the same phase space, it is possible to increase the circulating beam brightness with respect to that of the incoming beam by a large factor. The stored protons pass repeatedly through the stripper, however, so that this phase space is gradually enlarged by scattering. In this paper we consider how the circulating beam phase space (emittance) growth rate depends on the nature of the scattering process and on where it occurs in the storage ring matrix. Since the motivation for this work arose in connection with the design of the proposed high-current storage ring at LAMPF, the results are focused on the specific parameters of that device. The formalism is developed with some generality, however, subject only to the following restrictions. The stripper thickness (thin foil) is assumed negligible in comparison with the mean betatron wavelength of the machine. Residual gas scattering is neglected (i.e. high vacuum in the ring). All inelastic and nuclear elastic scattering is assumed negligible in comparison with Coulomb elastic scattering for the small angles considered here. Finally, we restrict ourselves to the case in which the circulating protons undergo no more than one Coulomb scattering per stripper traversal.

Derivation of Scattering Distributions in Phase and Betatron Space

We consider a hypothetical storage ring with a stripper foil at some location s=0 (= C = 2C = 3C, etc. where C is the ring circumference). We follow the betatron motion of a large number of particles injected into the ring with displacement y_0' from the equilibrium orbit and with inclination y_0'. By y we mean either radial or vertical displacement. We employ a graphical phase plane analysis analogous to that of Bruck[2] to deduce the distribution of betatron amplitudes after the particles have passed through the foil a large number of times, N.

The general form of motion of a non-equilibrium particle is

$$y = \hat{y}(\beta/\beta_0)^{1/2} \cos(\phi-\delta), \qquad (1)$$

where y, β, ϕ, are all functions of location s. β is the betatron function of the ring, and $\phi=\int ds/\beta$; β_0 is the value of β at the foil location, s=0. The constants $\hat{y}$ (betatron amplitude) and δ are determined from the initial values y_0 and y_0'. The slope of the trajectory is obtained by differentiating Eq. (1), using the relation $\phi' = 1/\beta$;

$$(\beta\beta_0)^{1/2}\, y' = -\hat{y}[\sin(\phi-\delta) - (\beta'/2)\cos(\phi-\delta)] \,. \qquad (2)$$

Solving Eqs. (1) and (2) for the sine and cosine:

$$\cos(\phi-\delta) = y/\left(\hat{y}(\beta/\beta_0)^{1/2}\right) \qquad (3)$$

$$\sin(\phi-\delta) = \left(\beta y' - (\beta'/2)y\right) / \left(\hat{y}(\beta/\beta_0)^{1/2}\right) \qquad (4)$$

Squaring and adding Eqs. (3) and (4), we obtain the Courant-Snyder invariant:

$$\frac{y^2 + [\beta y' - (\beta'/2)y]^2}{\beta} = \frac{\hat{y}^2}{\beta_0} = A \qquad (5)$$

It can be readily shown that the (y,y') phase space area occupied by all particles whose betatron amplitudes are less than or equal to y is πA, a fact which we shall use later.

Equations (3) and (4) can be depicted graphically on a phasor diagram (such as used in electrical circuit theory) as follows: envision a vector, attached at the origin of a two-dimensional space, with amplitude $\hat{y}(\beta/\beta_0)^{1/2}$, and making a negative angle (ϕ-δ) with the horizontal axis. Then the horizontal projection of the vector is y, while the vertical projection is $\beta y' - (\beta'/2)\, y \equiv y^*$.

Since we are considering only scattering at a fixed s location (the foil), the phasor representing unperturbed motion has constant amplitude $\hat{y} = [y_0^2 + (\beta_0 y_0' - (\beta_0'/2)y_0)^2]^{1/2}$, and advances clockwise, as the proton makes each circuit of the ring, in angular steps of $\Delta\phi = \int_0^C ds/\beta = 2\pi\nu$ radians, where ν (the tune of the motion) is the number of betatron oscillations in one revolution. A collision with a foil atom produces a small change $\delta\vec{y}$ in the vector representing betatron motion. This incremental vector has only a vertical component, $\delta y^* = \beta_0 \delta y'$, since the trajectory slope is changed but not the trajectory location. If the tune, ν, is not numerically the ratio of small integers (i.e. not at a resonance) then the mean value of 1/2 for the square of the cosine or the sine of the betatron phase at collision can be employed exactly as in Bruck's Eq. (14.36), and the incremental vectors can be added to the unperturbed vectors exactly as in his analysis, to give the probability distribution function for observing a particle with initial parameters y_0, y_0' to have parameters y, y^* after N turns. This function is, where $\sigma_N^2 = \frac{1}{2} N_s \beta_0^2 (\delta y')^2$,

$$P_N(y,y^*) = \exp\left(-(\vec{y}-\vec{y}_{0r})^2/2\sigma_N^2\right)/2\pi\sigma_N \,, \qquad (6)$$

(cf. Bruck's Eqs. (14.35) and (14.36) with $\lambda = \beta_0$).

N_s is the number of scatterings in N circuits of the ring, which in terms of foil thickness t, atom density n, and Coulomb scattering cross section σ_c is $N_s = Nnt\sigma_c$. The vector $\vec{y}$ is the vector (in the y,y* plane) representing a particular betatron motion, while $\vec{y}_{0r}$ is the vector with components $(y_0, \beta_0 y'_0 - (\beta'_0/2)y_0)$, rotated clockwise through an angle N2πν.

Transformed into the conventional y,y' phase plane, the probability distribution becomes

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$$P_N(y,y') = \left(\beta_0/2\pi\sigma_N^2\right)\exp\left\{-\left[(\delta y)^2 + \left(\beta_0\delta y' - \left(\beta_0'/2\right)\delta y\right)^2\right]/2\sigma_N^2\right\}, \quad (7)$$

where $\delta y = y - y_{or}$ and $\delta y' = y' - y'_{or}$ with y_{or} and y'_{or} being the mapping of the point (y_{or}, y^*_{or}) onto the y,y' plane.

This last result can be interpreted as follows: If many particles are injected into the ring with essentially the same initial conditions, then after N turns the particles will be distributed in phase space about the y,y' coordinates of an unperturbed orbit such that the contours of equal probability are ellipses similar to the betatron motion ellipse at $s = 0$. In particular, if the initial conditions are $y_0 = 0 = y_0'$, then the contours of equal probability in phase space are given by

$$y^2 + \left(\beta_0 y' - \left(\beta_0'/2\right)y\right)^2 = \text{constant.}$$

Further, integrating Eq. (7) over all y' yields the result that the beam is distributed in space in a Gaussian (normal) profile centered on the unperturbed orbit location.

If the injected beam has a normalized distribution function in y,y^* space given by

$$f_0(y,y^*) = \exp\left(-\left(y^2+y^{*2}\right)/2\sigma_0^2\right)/2\pi\sigma_0^2 \quad (8)$$

which we take as defining a "matched" beam, then after N turns the distribution will be given by

$$f_N(y,y^*) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_0(y,y^*)P_N(y-\bar{y},y^*-\bar{y}^*)d\bar{y}d\bar{y}^* , \quad (9)$$

which is readily evaluated to be

$$f_N(y,y^*) = \exp\left(-\left(\hat{y}^2+y^{*2}\right)/2\left(\sigma_0^2+\sigma_N^2\right)\right)/2\pi\left(\sigma_0^2+\sigma_N^2\right). \quad (10)$$

Thus the foil scattering width σ_N adds quadratically to the injected beam width σ_0.

The betatron amplitude distribution function can be obtained from this last result as (cf. Ref. 2, Eq. (14.42)):

$$p(\hat{y})d\hat{y} = \hat{y}\exp\left(-\hat{y}^2/2\left(\sigma_0^2+\sigma_N^2\right)\right)/\left(\sigma_0^2+\sigma_N^2\right)d\hat{y} , \quad (11)$$

which is a Rayleigh distribution. This distribution yields a mean square betatron amplitude $\overline{\hat{y}^2} = 2\left(\sigma_0^2+\sigma_N^2\right)$.

Emittance Growth for Summed Distributions

If we inject continuously for N turns, particles injected in the first turn will have undergone N foil traversals, those injected in the 2nd turn N-1 traversals, and so on. The betatron amplitude distribution (normalized) representing the scattering history of all particles accumulated in the ring is then the sum:

$$P_{\Sigma N} = \frac{1}{N}\sum_{i=1}^{N}\hat{y}\exp\left(-\hat{y}^2/2\left(\sigma_0^2+i\delta\sigma^2\right)\right)/\left(\sigma_0^2+i\delta\sigma^2\right) \quad (12)$$

where $\delta\sigma^2 = 1/2\beta_0^2 nt\sigma_c\overline{(\delta y')^2}$ is the incremental increase in the squared scattering width added per foil traversal. While this distribution cannot be evaluated explicitly in general, its mean square value is defined and is just:

$$\overline{\hat{y}_\Sigma^2} = 2\left[\sigma_0^2+1/2(N+1)\delta\sigma^2\right] \simeq 2\left[\sigma_0^2+1/2\sigma_N^2\right] , \quad (13)$$

the approximation being valid for large N.

If $i\delta\sigma^2$ is small in comparison with σ_0^2, and N is not too large, it can be shown numerically that the sum $P_{\Sigma N}(y)$ is reasonably well approximated by the Rayleigh distribution:

$$p(\hat{y}) = \hat{y}\exp\left(-\hat{y}^2/2\left(\sigma_0^2+1/2\sigma_N^2\right)\right)/\left(\sigma_0^2+1/2\sigma_N^2\right) \quad (14)$$

Using the relation between the Courant-Snyder invariant and the stored beam emittance, it is seen that the (y,y') phase space occupied by particles with betatron amplitudes less than or equal to the RMS value $(\overline{\hat{y}^2})^{1/2}$ is:

$$\varepsilon_N = \pi\overline{\hat{y}^2}/\beta_0 = (2\pi/\beta_0)(\sigma_0^2+1/2\sigma_N^2) \quad (15)$$

$$= \varepsilon_0 + \pi N\delta\sigma^2/\beta_0 = \varepsilon_0 + 1/2\pi\beta_0 Nnt\sigma_c\overline{(\delta y')^2}$$

where ε_0 is the initial emittance of the injected H^- beam. Thus we observe that the emittance of the stored beam is proportional to the value of the betatron function at the stripper location, the areal density of the target, the Coulomb scattering cross section of the target atoms, and the mean square scattering angle produced in the collisional process, and that it grows linearly with the number of injected turns. In Fig. 1 we have plotted the scattering width σ_N versus the product Nnt, with β_0 as a parameter.

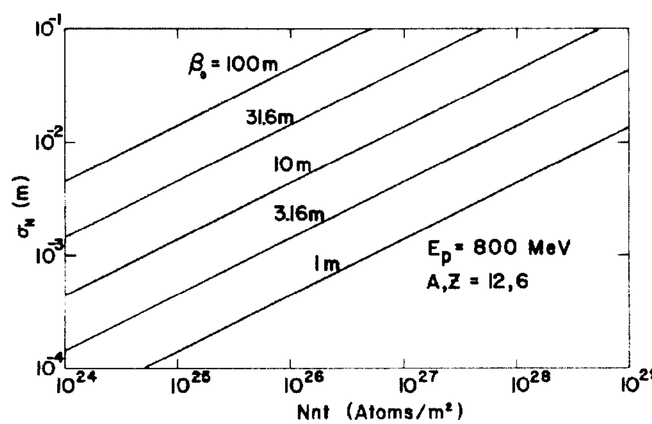

Figure 1

Scattering Width σ_N Versus Nnt, with β_0 as a Parameter. E_p = 800 MeV. Stripper Material - Carbon.

The energy of the incident protons was taken as 800 MeV, the stripper material as carbon (A,Z = 12,6), and the product $\sigma_c\overline{(\delta y')^2} = 3.84 \times 10^{-33}$ (m-rad)^2/atom was obtained from Bruck (Eq. (14.10)), corrected for relativistic effects and center-of-mass motion.

The stored beam emittance growth (Eq. (15)), under conditions pertinent to the LAMPF storage ring proposal, is plotted in Fig. 2 as a function of number of

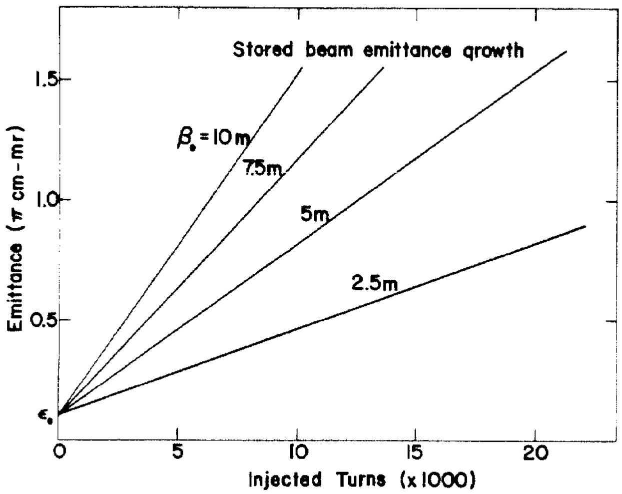

Figure 2

Stored Beam Emittance Growth Versus Number of Injected Turns, with β_0 as a parameter. E_p = 800 MeV. Stripper = 150 $\mu g/cm^2$ carbon foil.

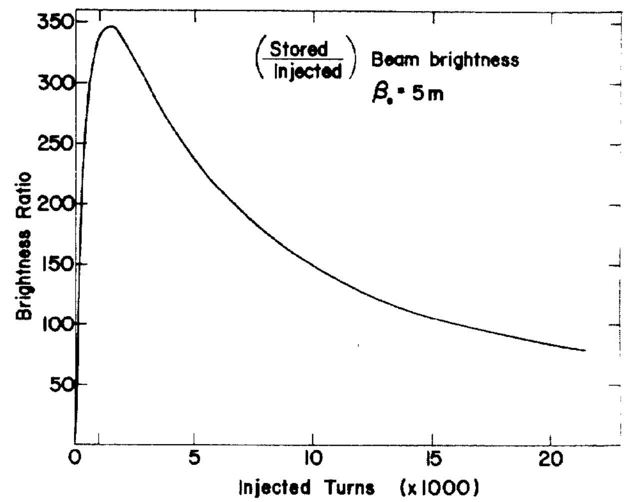

Figure 3

Ratio of Stored to Injected Beam Brightness Versus Number of Injected Turns, for β_0 = 5 m, and ε_0 = (π/10) cm-mrad

injected turns, with β_0 as a parameter. The emittance ε_0, for the injected H^- beam is taken as (π/10) x 10^{-5} m-rad. Using the recently measured 800 MeV carbon stripping cross sections (σ_{-10} and σ_{01})[3], and choosing 95% $H^- \rightarrow H^+$ conversion to be an acceptable efficiency, we arrive at nt = 7.5 x 10^{22} atoms/m^2. $\sigma_c(\delta y')^2$ has the same value as for Fig. 1. It is apparent from the figure that (depending on the choice of β_0) we should be able to inject from 10,000 to 20,000 turns without increasing the stored beam emittance to an unacceptably large value.

Brightness Ratio

It is interesting to examine the dependence of the stored beam brightness on the number of turns injected into the ring. If brightness is defined to be proportional to beam current and inversely proportional to the product of the emittances in y,y' and x,x' phase space, $(B \propto i/E_x E_y)$, then the ratio between the brightness of the stored beam and that of the injected beam, after N turns of continuous injection is:

$$R_B = B_N/B_o = (Ni/\varepsilon_{Nx}\varepsilon_{Ny})/(i/\varepsilon_{ox}\varepsilon_{oy})$$
$$= N/\left[\left(1+1/2N\delta\sigma_x^2/\sigma_{ox}^2\right)\left(1+1/2N\delta\sigma_y^2/\sigma_{oy}^2\right)\right] .$$

If we assume equal β_0 and ε_0 in the x,x' and y,y' planes, then this simplifies to

$$R_B = N/\left(1+1/2N\delta\sigma^2/\sigma_0^2\right)^2$$

This function is plotted in Fig. 3 for β_0 = 5 m, and all other conditions as specified in Fig. 2. It can be seen that the beam brightness ratio rises rapidly to a maximum of nearly 350 at N = 1500 turns, and then drops off rather slowly. Even at N = 20,000, the circulating beam is more than 80 times brighter than the injected beam.

References

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