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ANALYTICAL AND COMPUTATIONAL STUDIES OF RESONANCE WIDTH NEAR THE INTERSECTION OF TWO RESONANCES

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Abstract

When the working point (ν_x,ν_y) lies near the intersection of two sum resonances that arise from the same nonlinear multipole field, the width of each resonance is modified by the presence of the other. Taking the two third-integer resonances $3\nu_x=N$ and $\nu_x+2\nu_y=N$ as an example, we have developed an analytical treatment of this problem and its predictions have been compared with numerical results.

Introduction

It is well known that a perturbation term in the Hamiltonian can lead to resonance between the sum or difference of integral multiples of the two transverse oscillation frequencies and a multiple of the revolution frequency. Standard methods are available[1] for determining the resonance band width in the case of an isolated resonance of this kind, and lead to the necessity of avoiding frequencies lying within an isolated resonance band.

Meier and Symon[2] have explored the region near the intersection of a sum and difference resonance in an effort to explain how the difference resonance (which by itself does not lead to unlimited growth of the oscillation amplitude) may cause actual instabilities. Their analysis starts with a single nonlinear perturbing term in the Hamiltonian. In our paper we consider two perturbing terms in the Hamiltonian which lead to two different sum resonances, and explore the region around the intersection of these resonances in order to determine whether the intersecting resonant bands are distorted appreciably.

Analytic Treatment

Our starting point is the Hamiltonian

$$H = (x'^2+\nu_x^2x^2+y'^2+\nu_y^2y^2)/2$$

$$-(ax^3/3-bxy^2)\cos(N\theta) \qquad (1)$$

corresponding to the coupled equations of motion

$$x''+\nu_x^2x = (ax^2-by^2)\cos(N\theta), \qquad (2a)$$

$$y''+\nu_y^2y = -2bxy\cos(N\theta). \qquad (2b)$$

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For the usual sextupole fields, one would have a=b, but we will keep a and b independent during the course of the analysis. A standard phase-amplitude method is used to analyze the resonance terms. Setting

$$x=P\sin(\nu_x\theta+\alpha),\quad y=Q\sin(\nu_y\theta+\beta) \qquad (3)$$

and averaging over all oscillatory terms not involving the two resonances

$$3\nu_x=N \qquad (4a)$$

and

$$\nu_x+2\nu_y=N, \qquad (4b)$$

we obtain

$$P'\simeq-(aP^2/8\nu_x)\cos\psi + (bQ^2/8\nu_x)\cos\chi \qquad (5)$$

$$Q'\simeq(bPQ/4\nu_y)\cos\chi \qquad (6)$$

$$\psi'\simeq\Delta_1 + (3aP/8\nu_x)\sin\psi$$
$$-(3bQ^2/8\nu_xP)\sin\chi \qquad (7)$$

$$\chi'\simeq\Delta_2 + (aP/8\nu_y)\sin\psi$$
$$-(b/8\nu_x\nu_yP)(4\nu_xP^2+\nu_yQ^2)\sin\chi \qquad (8)$$

where

$$\psi = \Delta_1\theta+3\alpha \qquad (9a)$$
$$\chi = \Delta_2\theta+\alpha+2\beta \qquad (9b)$$
$$\Delta_1 = 3\nu_x-N \qquad (10a)$$
$$\Delta_2 = \nu_x+2\nu_y-N \qquad (10b)$$

It is possible to construct a new Hamiltonian in the variables P^2, Q^2, ψ and $(3\chi-\psi)/2$, which corresponds to the equations of motion (5) - (8), but we have been unable to determine the modified resonance bands from the fact that this Hamiltonian is constant. Instead, we have alternately treated the term in b, and the term in a, as small, and from this have determined the way in which each resonance affects the other as we approach the intersection from afar. This is equivalent to determining the border of stability Δ_1 for the resonance $3\nu_x=N$ up to and including terms in $(1/\Delta_2)$, and the border of stability Δ_2 for the resonance $\nu_x+2\nu_y=N$ up to and including terms in $(1/\Delta_1)$.

The analysis proceeds from (5) - (8) by first considering the $\sin\chi$ and $\cos\chi$ terms to be rapidly varying compared to the changes in the variables P and ψ. In this way one separates all terms into slowly varying and rapidly varying parts in the form

$$P \simeq P_o + P_1 \sin\chi_o \quad (11a)$$

$$Q \simeq Q_o + Q_1 \sin\chi_o \quad (11b)$$

$$\psi \simeq \psi_o + \psi_1 \cos\chi_o \quad (11c)$$

$$\chi \simeq \chi_o + \chi_1 \cos\chi_o \quad (11d)$$

where P_1, Q_1, ψ_1 and χ_1 are obtained from (5), (6), (7) and (8), and are proportional to $1/\Delta_2$. Averaging over χ_o, one obtains coupled differential equations for P_o and ψ_o, for which an integral of the motion exists. A study of the corresponding trajectories then leads to the conclusion that the motion will be bounded and stable if the initial value of P_o (allowing all initial values of ψ_o) is less than a particular value which depends on a, b, Δ_1 and Δ_2. Taking into account the arbitrary initial value of χ_o, one converts the condition on the initial value of P_o to one on the initial value of P_1 using (11a). This ultimately leads to the two borders of the $3\nu_x=N$ band being given by

$$\Delta_1 \simeq \pm(3aP/4\nu) + (3b^2Q^2/32\nu^2) \times [2\pm(a/b)](1/\Delta_2) \quad (12)$$

where P and Q are the initial amplitudes (all initial phases α and β are permitted) and where we have set $\nu_x \simeq \nu_y \simeq \nu$ in all terms where their difference is unimportant. A similar but more complicated expression can be obtained for the dependence of the borders of the Δ_2 resonance band on the presence of a high frequency term in ψ. This leads to

$$\Delta_2 \simeq \pm(b/2\nu)(P+\sqrt{2}\,Q) + (a^2/128)^2) \times [4P^2+2\sqrt{2}\,PQ+Q^2) \pm (8bP^2/a)](1/\Delta_1). \quad (13)$$

It is not clear whether the averaging process is reliable when the two oscillating terms are comparable in frequency. For this reason, we have performed numerical computations using impulse representations of the nonlinear terms in the Hamiltonian.

Numerical Results

The model used for the numerical computation is identical to the one described by Meier and Symon.[2] Nonlinear terms in the Hamiltonian are simulated by δ-function impulses. There is no change in x or y across each impulse but the changes in x' and y' are proportional to ax^2-by^2 and $-2bxy$, respectively. Two-by-two matrices are used to transform (x,x') and (y,y') from one impulse to the next. Since a single δ-function impulse per revolution contains all harmonic components, resonating as well as non-resonating, four impulses of alternating signs are used,

$$-\delta(\theta),\ \delta(\theta-\pi/2),\ -\delta(\theta-\pi)$$

$$\text{and } \delta(\theta-3\pi/2). \quad (14)$$

Resulting harmonic components are of the form

$$\cos[(4n+2)\theta];\ n=0,1,2,\ldots.. \quad (15)$$

so that only one harmonic component should be dominant for resonances under consideration. Results presented here are for N=2, that is, for $3\nu_x=2$ and $\nu_x+2\nu_y=2$. The strength of the impulse is chosen such that

$$a = (2/\pi)\nu_x^{3/2}; \quad b = (2/\pi)\nu_x^{1/2}\nu_y. \quad (16)$$

As the initial amplitude of the oscillation, we have taken

$$P(\theta=0) = (1.5557\times10^{-4}/\nu_x)^{1/2}, \quad (17a)$$

$$Q(\theta=0) = (1.5557\times10^{-4}/\nu_y)^{1/2}. \quad (17b)$$

Since we are interested in the boundary of the region in (ν_x,ν_y) space inside of which all particles remain stable regardless of their initial phase (α,β), we have generally chosen three "worst" sets of (α,β) such that

$$\sin(3\alpha) = 1 \text{ for } \Delta_1>0,$$
$$= -1 \text{ for } \Delta_1<0$$

and

$$\sin(\alpha+2\beta) = -1 \text{ for } \Delta_2>0,$$
$$= 1 \text{ for } \Delta_2<0.$$

For example, with $\Delta_1>0$ and $\Delta_2>0$, (α,β) = (30°, 120°), (150°, 60°) and (270°, 0°). Note that a change of the value of β by 180° merely changes the signs of y and y' with no change in the resonance boundaries. For a few combinations of Δ_1 and Δ_2, we have investigated all combinations of α=0°, 30°, ..., 330° and β=0°, 30°, ...,150° to check that the above three sets indeed give the boundary. Particles are regarded as "unstable" if their amplitude, either P or Q, grows by a factor three or more compared to the original value in less than 10,000 revolutions. Here we are not dealing with long-term beam instabilities like Arnold diffusion[3] and 10,000 revolutions should be adequate for our purpose.

All numerical calculations have been performed with the CDC-6600 (single precision) and PDP-10 (double precision) at Fermilab. In Table 1, the change in the resonance width of $3\nu_x=2$ due to the presence of the resonance $\nu_x+2\nu_y=2$ is tabulated as a function of the quantity $1/|\Delta_2|$. The effect of $3\nu_x=2$ on the resonance width of $\nu_x+2\nu_y=2$ is given in Table 2. Plus and minus values indicate, respectively, an increase and a decrease in the width. Although the analysis given in the previous section is applicable only when $|\Delta_2|$

is much larger than or much smaller than $|\Delta_1|$, the change in the boundary near the center ($|\Delta_1| \approx |\Delta_2|$) is presented in Table 3 to complete the picture. The overall shape of the distortion in the resonance boundaries is illustrated in Fig. 1 where the magnitude of the change is intentionally exaggerated in some areas in order to show the qualitative features. Note that there is a distortion in the resonance center lines as well as in the boundaries.

Discussion

As shown in Table 1, the change in the resonance width of $3\nu_x=2$ due to the presence of the resonance $\nu_x+2\nu_y=2$ is predicted very well by the analytical treatment presented in this paper when $|\Delta_2|$ is not too small. On the other hand, the effect of $3\nu_x=2$ on the resonance boundary of $\nu +2\nu_y=2$ seems to be more complicated and our analysis here is not entirely satisfactory. (See Table 2.) One difficulty in the comparison of numerical results with the theoretical prediction is the impossibility of avoiding non-resonating harmonic components (6θ, 10θ, etc. here) when δ-function impulses are used. We have tried to minimize this effect by taking four impulses of alternating signs instead of a single impulse. The action of the undesirable harmonic components is seen in the unperturbed values of Δ_1 and Δ_2. This is particularly serious when one tries to predict the change in Δ_2 by the resonance $3\nu_x=2$. The unperturbed value of Δ_2 itself already differs from the theoretical value by as much as $(30\sim40)\times10^{-6}$, a value comparable with or even larger than the predicted magnitude of the change. This may explain a somewhat better agreement one can get when one takes the average change for $\Delta_2>0$ and $\Delta_2<0$ (the third and the fourth columns, respectively, in Table 2).

Results presented here clearly demonstrate that, when two (or more) resonances intersect each other in (ν_x, ν_y) plane, there is a sizable distortion in the resonance width and in the central line of the resonance. It may be necessary to take this into account when, for example, one tries to find the optimum working line in the tune diagram for a storage ring.

References

1. See, for example, R. Hagedorn, CERN Report 57-1, March 7, 1957; P.A. Sturrock, Annals of Physics 3, 113 (1958).

2. H. Meier and K.R. Symon, Proceedings of the International Conference on High-Energy Accelerators and Instrumentation, CERN, September 1959, p. 253.

3. V.I. Arnol'd, Dokl. Akad. Nauk. SSR 156 9(1964); L.C. Teng, Performance Limitations Imposed by Beam Dynamics, to be published in this issue.

Table 1 Change in the resonance width of $3\nu_x=2$ due to $\nu_x+2\nu_y=2$. All values except $1/|\Delta_2|$ are in 10^{-6}. Unperturbed widths are $\Delta_1=0.005940$ and -0.005978 Plus (minus) values indicate an increase (decrease) in the width.

(A) $\Delta_1 \cdot \Delta_2 >0$

$1/\lvert\Delta_2\rvert$	theory	numerical $\Delta_1>0$	numerical $\Delta_1<0$
5	89	96	91
10	177	195	188
20	355	413	402
30	532	667	651
40	709	971	948
50	887	1346	1315

(B) $\Delta_1 \cdot \Delta_2 <0$

$1/\lvert\Delta_2\rvert$	theory	numerical $\Delta_1>0$	numerical $\Delta_1<0$
5	-30	-28	-34
10	-59	-61	-65
20	-118	-127	-129
30	-177	-200	-198
40	-236	-282	-276
50	-296	-379	-366

Table 2 Change in the resonance width of $\nu_x+2\nu_y=2$ due to $3\nu_x=2$. All values except $1/|\Delta_1|$ are in 10^{-6}. Unperturbed widths are $\Delta_2=0.009553$ (with $\Delta_1>0$) and 0.009572 (with $\Delta_1<0$); $\Delta_2=-0.009624$ (with $\Delta_1>0$) and -0.009614 (with $\Delta_1<0$). Plus (minus) value indicate an increase (decrease) in the width.

(A) $\Delta_1 \cdot \Delta_2 >0$

$1/\lvert\Delta_1\rvert$	theory	numerical $\Delta_2>0$	numerical $\Delta_2<0$
5	39	69	22
10	78	111	62
20	156	201	150
30	234	300	247
40	312	411	363
50	390	416	370

(B) $\Delta_1 \cdot \Delta_2 <0$

$1/\lvert\Delta_1\rvert$	theory	numerical $\Delta_2>0$	numerical $\Delta_2<0$
5	0	14	-25
10	1	15	-22
20	2	21	-14
30	3	29	-4
40	3	41	10
50	4	56	27

Table 3 Resonance boundaries near the intersection of $3\nu_x=2$ and $\nu_x+2\nu_y=2$. In (ν_x,ν_y) space, $\nu_x=2/3+\Delta_1/3$ and $\nu_y=2/3+(3\Delta_2-\Delta_1)/6$.

$\Delta_1\cdot\Delta_2>0$		$\Delta_1\cdot\Delta_2<0$	
Δ_1	Δ_2	Δ_1	Δ_2
.01866	.01	.014	-.009712
-.01895	-.01	-.014	.009685
.017	.01016	.012	-.009751
-.017	-.01026	-.012	.009720
.008185	.015	.01	-.009855
-.008182	-.015	-.01	.009803
.01194	.01194	.009170	-.01
-.01188	-.01188	-.007979	.01
		.005346	-.015
		-.005410	.015
		.005063	-.0125
		-.005150	.0125
		.004852	-.01111
		-.004974	.01111

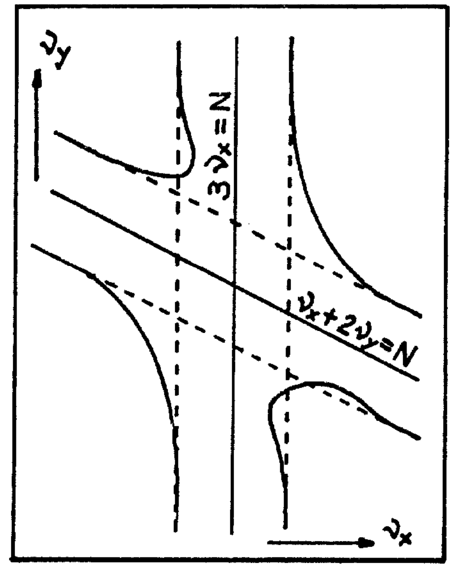

Fig. 1 Distortion in the resonance boundaries when two resonances $3\nu_x=N$ and $\nu_x+2\nu_y=N$ coexist. The magnitude of the distortion is intentionally exaggerated in some areas in order to clearly show the qualitative features.