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EFFECTS OF BEAM-BEAM FORCES IN LARGE ELECTRON-POSITRON STORAGE RINGS

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1. Introduction

A symmetrical arrangement of sextupole magnets is included in the design of large electron-positron storage rings so as to provide low values of chromaticity. The strength of the sextupoles is such that they contribute appreciable non-linear effects. The maximum number of particles that can be brought into collision in the rings is then determined by the combined effects on particle motion of the sextupole fields and the beam-beam space charge forces.

Electrons and positrons collide at a few interaction regions where there are linear and non-linear components of the space charge forces between the colliding beams. The non-linear components excite non-linear resonances, while the linear components may be considered to introduce a modified β-function throughout the magnet lattice. Such a modified β-profile alters the non-linear contributions of the lattice sextupoles, and distributions of sextupoles which are optimised prior to beams being brought into collision are, in general, not optimum for the collision mode. Also, off-momentum particles travel off-axis through the sextupoles which adds an equivalent quadrupole effect. The oscillation of an off-momentum particle may be analysed by assuming that the quadrupole term gives a further change in the β-function, with consequent effect on the non-linear motion.

Approximate methods are described for estimating the resonance widths due to the combined action of the sextupoles and the beam-beam forces. In particular, the fourth-order terms due to the sextupoles are obtained, both the resonance terms and those terms which give the variation of tune with amplitude. This work was initiated to assess acceptable sextupole distributions, to act as a guide in interpreting detailed tracking programs and to obtain estimates of possible beam-beam limits. A more accurate treatment would extend the analysis to include betatron-synchrotron coupling.

2. Non-Linear Resonance Theory

R Hagedorn[1] is one of a number of authors who has analysed particle motion in accelerators under the presence of non-linear magnetic fields. He gives a general analysis of coupled transverse betatron motion, using a co-ordinate system (x, y, θ) where particle azimuth is defined by the angle θ and the plane (x, y) is orthogonal to the direction of motion. In this co-ordinate system, 2n-pole fields have complex Fourier components of potential of the form $x^{(n-q)}\, y^q\, e^{ip\theta}$, where n, q and p are integers. Hagedorn first derives n^{th} order Hamiltonians to describe the non-linear motion that results from the 2n-pole fields. Resonance is defined by $(n-q)\, Q_h + q\, Q_v = p$, where Q_h and Q_v are betatron tunes. For the particular case of sextupoles, where n = 3, Hagedorn continues the analysis to the next approximation and finds the sextupole contributions to the fourth-order Hamiltonians. Such terms are significant for the sextupole distributions required in large e^+-e^- storage rings.

In a later report, G Guignard[2] obtains a more convenient expression for the n^{th} order Hamiltonians of the 2n-pole fields. These are given as a function of the lattice β-values and the azimuthal distribution of the 2n-pole elements. The nomenclature used for the n^{th} order Hamiltonian coefficients is $h^{(n)}_{jklm(p)}$ where $n = (j+k+l+m)$ and $q = (l+m)$. Guignard derives widths of the sum resonances in terms of these h coefficients. Expressions for third order resonances are given in section 3.

Examples of the nomenclature are given for two resonance lines shown in Figure 3, which is drawn for Q values near 19.15, a possible operating point for the proposed storage ring, EPIC:

$$h^{(3)}_{0120(20)} \quad \text{for} \quad 2Q_v - Q_h = 20, \text{ and}$$

$$h^{(4)}_{2020(76)} \quad \text{for} \quad 2Q_v + 2Q_h = 76.$$

The contribution to fourth-order Hamiltonians that arises from sextupole fields is given by Hagedorn, as infinite sums of products of pairs of the third-order coefficients. Examples of these infinite sums are given in section 4 for sum and difference resonances. The following terms give the variations of Q_v and Q_h as a function of the oscillation amplitudes:

$$g^{(4)}_{2200(0)}, \quad g^{(4)}_{0022(0)}, \quad g^{(4)}_{1111(0)}$$

As the infinite sums involve products of pairs of the third-order coefficients, the effect of an individual sextupole depends not only on its magnitude and on the local β-functions but on the strengths of all the sextupoles in the storage ring and on the β-values at each sextupole.

Space charge fields may also be analysed for resonance effects by deriving the relevant non-linear terms. This approach has been adopted by B W Montague[3] E Keil[4] and A G Ruggiero[5]. In the case of head-on e^+-e^- collisions, with Gaussian particle distributions in the x, y directions, the space charge potential may be expanded as an infinite series of terms with even-order powers of x and y. The lowest order terms involve x^2 or y^2 and lead to linear focussing in the x and y directions at each interaction region. The linear terms may be separated from the higher order terms and may be considered as introducing a β-change through the lattice. In particular, the minimum β-value at the interaction point, β^* will be perturbed to some new value, β^{**}:

$$\beta^{**} = \beta^*/[1 - \eta^2 + 2\eta \cot \mu]^{1/2}$$

where μ is the phase shift of the betatron motion per superperiod, and

$$\eta = 0.5\, \beta^*/F$$

with F the equivalent focal length of the linear beam-beam lens. Any given non-linear space charge term will contribute to a particular resonance, and the resonance width is obtained from the appropriate Guignard integral. Since there is only one interaction point per superperiod, the integral is readily evaluated. The correct β-values to use in the integral are the β^{**} values given above. A typical resonance due to non-linear space charge terms is evaluated in section 5. For the motion of an off-momentum particle, there is an additional change in β^{**} due to the action of the chromaticity-correcting sextupoles.

3. Examples of Third-Order Hamiltonian Coefficients

Again refer to Figure 3 for examples of resonance lines. The operating Q-point is shown near 19.15, though alternative points exist near 18.15 and 15.15. With four superperiods in EPIC, there are five neighbouring, systematic, third-order resonances which may be excited by the lattice sextupoles. There are no equivalent third-order terms in the beam-beam space charge forces in the particular case of head-on collisions. The coefficients that describe these five systematic third-order resonances are:

$$h^{(3)}_{3000(56)},\ h^{(3)}_{3000(60)},\ h^{(3)}_{1020(56)},\ h^{(3)}_{1020(60)},\ h^{(3)}_{0120(20)}.$$

General formulae for Hamiltonians and resonance widths are given in reference (2). These are defined in terms of the parameters:

R,	the mean radius of the storage ring,
s,	the length along the orbit measured from the interaction point,
β_h,β_v,	the lattice β-functions at the positions of the non-linear elements,
$\varepsilon_h,\varepsilon_v$,	the horizontal and vertical beam emittances, and
K,	the parameter defining the strength of the non-linear element. For sextupoles, a normalised strength is defined with $K = \frac{\partial^2 B}{\partial x^2}/B\rho$.

Resonance widths for third-order sum resonances are given by:

$$\Delta e \ (\text{for } 3Q_h=p) \quad = \quad 18\sqrt{R\varepsilon_h}\ |\ h^{(3)}_{3000(p)}\ |$$

$$\Delta e \ (\text{for } 2Q_v+Q_h=p) \quad = \quad 8\left(1+\frac{\varepsilon_v}{4\varepsilon_h}\right)\sqrt{R\varepsilon_h}\ |\ h^{(3)}_{1020(p)}\ |$$

Examples of third-order coefficients are given for $h^{(3)}_{3000(56)}$ and $h^{(3)}_{1020(60)}$:

$$\int_{-\pi R}^{+\pi R}\left[-\frac{i\beta_h\sqrt{\beta_h R}\ K}{48\pi}\exp\left(i\int_0^s\left[\frac{3Q_h}{R}-\frac{3}{\beta_h}\right]ds\right)\exp\left(\frac{-56is}{R}\right)\right]\cdot\frac{ds}{R} = h^{(3)}_{3000(56)}$$

$$\int_{-\pi R}^{+\pi R}\left[\frac{i\beta_v\sqrt{\beta_v R}\ K}{16\pi}\exp\left(i\int_0^s\left[\frac{Q_h}{R}-\frac{1}{\beta_h}+\frac{2Q_v}{R}-\frac{2}{\beta_v}\right]ds\right)\exp\left(\frac{-60is}{R}\right)\right]\cdot\frac{ds}{R} = h^{(3)}_{1020(60)}$$

Many further third-order coefficients must be evaluated in order to obtain the fourth-order Hamiltonians which describe the fourth-order effect of the sextupoles. These include:

$$h^{(3)}_{3000(p)},\ h^{(3)}_{1020(p)},\ h^{(3)}_{0120(p)},\ h^{(3)}_{2100(p)},\ h^{(3)}_{1011(p)}$$

where p is any integer divisible by the superperiod number. The terms $h^{(3)}_{2100(p)}$ and $h^{(3)}_{1011(p)}$ are not resonances of third order, yet contribute to fourth-order resonances.

4. Fourth-Order Hamiltonians Due To Sextupoles

The coefficients of interest for the resonance diagram of Figure 3 are:

$$g^{(4)}_{2200(0)},\ g^{(4)}_{0022(0)},\ g^{(4)}_{1111(0)},\ g^{(4)}_{4000(76)},\ g^{(4)}_{0040(76)},$$

$$g^{(4)}_{2020(76)},\ g^{(4)}_{2002(0)}.$$

All but the last of these is derived in the form of infinite sums in reference (1), eg

$$g^{(4)}_{4000(76)} = h^{(4)}_{4000(76)} + 3i\sum_p \frac{h^{(3)}_{2100(p)}\ h^{(3)}_{3000(76-p)}}{(Q_h-p)}$$

The first term arises from any octupoles present, while the summation term is for the third-order coefficients due to sextupoles. An additional summation term is required if skew sextupole fields are present.

Resonance widths for the sum resonances are:

$$\Delta_e \ (\text{for } 4Q_h=p) \quad = \quad 32\ R\varepsilon_h\ |\ g^{(4)}_{4000(p)}\ |$$

$$\Delta_e \ (\text{for } 2Q_v+2Q_h=p) \quad = \quad 8\ \left(1+\frac{\varepsilon_v}{\varepsilon_h}\right) R\varepsilon_h\ |g^{(4)}_{2020(p)}|$$

The variation of Q_v and Q_h as a function of the beam emittances is:-

$$\delta Q_v \quad = \quad iR\ [\varepsilon_h\ g^{(4)}_{1111(0)} \ + \ 2\varepsilon_v\ g^{(4)}_{0022(0)}]$$

$$\delta Q_h \quad = \quad iR\ [\varepsilon_v\ g^{(4)}_{1111(0)} \ + \ 2\varepsilon_h\ g^{(4)}_{2200(0)}]$$

Finally, there is the difference resonance $2Q_v-2Q_h=0$, which is not derived in reference (1). This resonance line lies along the leading diagonal of Figure 3. It may be excited both by the lattice sextupoles and by the $x^2\cdot y^2$ component of the beam-beam space charge potential. The contribution of the lattice sextupoles is:

$$g^{(4)}_{2002(0)} = i\sum_p\left(\frac{h^{(3)}_{1002(-p)}\,[h^{(3)}_{2100(p)}+2h^{(3)}_{1011(p)}]}{(Q_h-p)} + \frac{3h^{(3)}_{3000(-p)}\ h^{(3)}_{0102(p)}}{-(Q_h+2Q_v+p)}\right)$$

5. Hamiltonians Due to Space Charge Fields

Space charge potentials have been derived in reference (3) for a beam of Gaussian distributions in both the x and y directions. An expansion of the potential is an infinite series of terms involving even order powers of x and y. The potential function may be used to obtain the angular deflection of a particle of one beam at an interaction region on passing a bunch of N particles of the second beam. In the x direction, the angular deflection for head-on e^+-e^- collisions is:

$$dx/ds \quad = \quad -\ 2Nr_eF(x,y)/\gamma\sigma_h(\sigma_h+\sigma_v)$$

where r_e is the classical electron radius ($= e^2/mc^2\ 4\pi\varepsilon_0$),

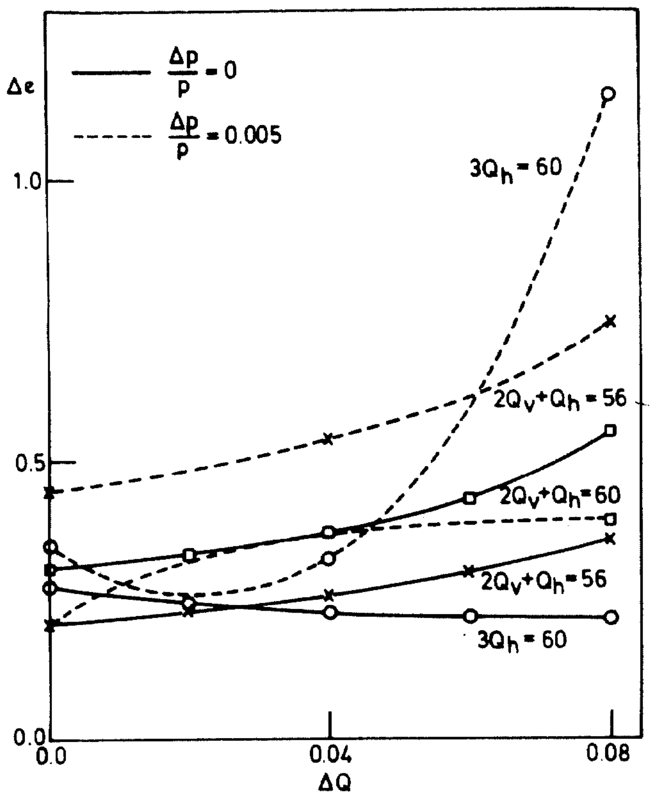

Fig. 1. Third Order Resonance Widths.

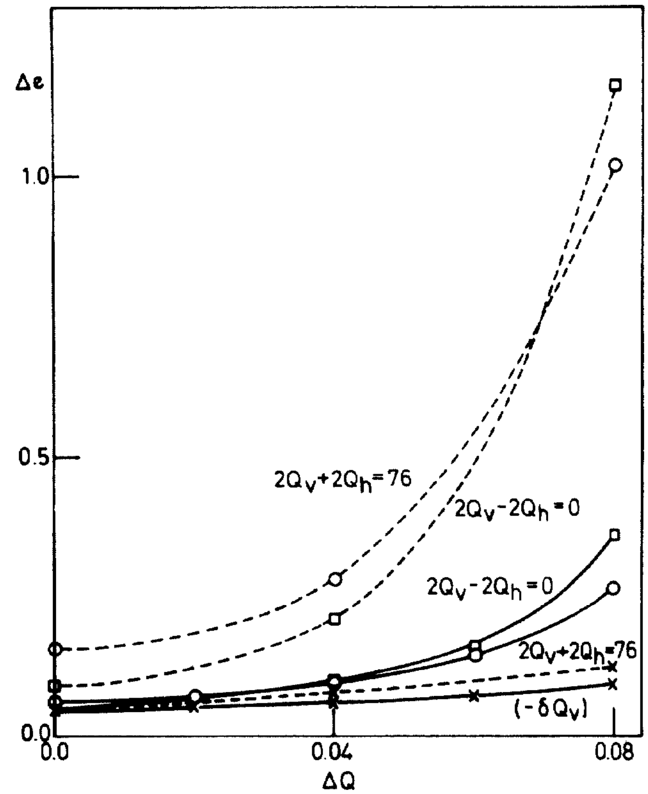

Fig. 2. Fourth Order Resonance Widths.

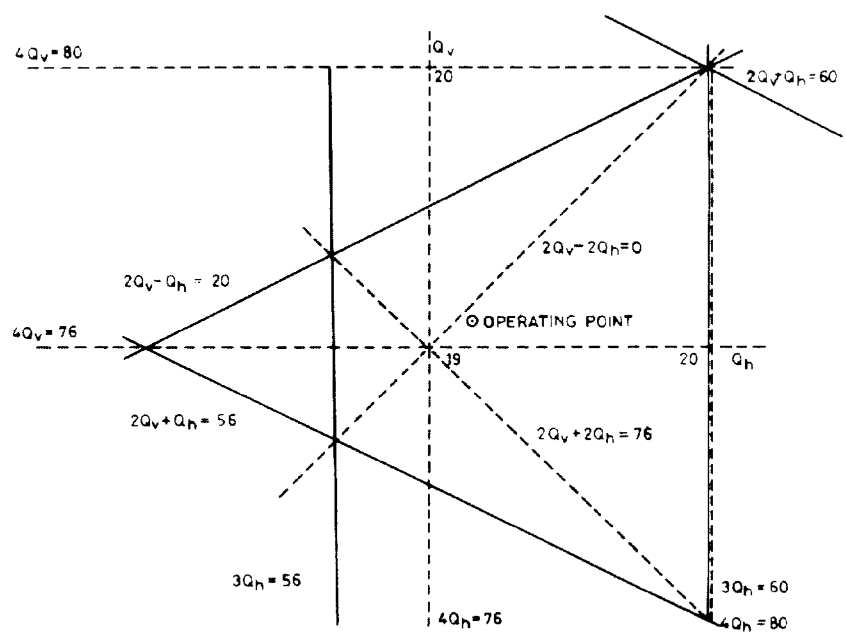

Fig. 3. Systematic 3rd and 4th Order Resonances for EPIC.

γ is the ratio of the particle energy to its rest energy,

σ_h, σ_v are standard deviations of the beam distributions at the interaction region in the x and y directions respectively, and

$$F(x,y) = x\,[1 - d_2 \left(\frac{x}{\sigma_h}\right)^2 - f_2 \left(\frac{y}{\sigma_v}\right)^2 + d_4 \left(\frac{x}{\sigma_h}\right)^4 + f_4 \left(\frac{y}{\sigma_v}\right)^2 \left(\frac{x}{\sigma_h}\right)^2 + h_4 \left(\frac{y}{\sigma_v}\right)^4 + \ldots]$$

$$d_2 = [2\sigma_h+\sigma_v]/6[\sigma_h+\sigma_v]$$

$$d_4 = [8\sigma_h^2+9\sigma_h\sigma_v+3\sigma_v^2]/120[\sigma_h+\sigma_v]^2$$

$$f_2 = \sigma_v/2\ [\sigma_h+\sigma_v]$$

$$f_4 = \sigma_v[3\sigma_h+\sigma_v]/12[\sigma_h+\sigma_v]^2$$

$$h_4 = \sigma_v[\sigma_h+3\sigma_v]/24[\sigma_h+\sigma_v]^2$$

The angular deflection in the y direction is obtained by interchanging all x and y in the above expressions together with interchanging all subscripts h and v.

A Hamiltonian coefficient due to the space charge field may be obtained from the formulae of Guignard after making a comparison of the space charge potential (or dx/ds, dy/ds) with the equivalent term due to a given non-linear thin magnetic lens. As an example, the $h^{(4)}_{4000(76)}$ coefficient is found to be:

$$h^{(4)}_{4000(76)} = -\frac{iM(\beta_h^{**})^2}{384\,\pi\,R}\ \frac{2Nr_e(2\sigma_h+\sigma_v)}{\gamma\,\sigma_h^3(\sigma_h+\sigma_v)^2}$$

where M is the number of interaction regions. The width of this resonance is given by:

$$\Delta e = \frac{M\varepsilon_h(\beta_h^{**})^2}{6\pi}\ \frac{Nr_e(2\sigma_h+\sigma_v)}{\gamma\,\sigma_h^3(\sigma_h+\sigma_v)^2}$$

6. Computational Methods and Results

Existing lattice programs have been expanded to include linear beam-beam focussing terms and to give Hamiltonian coefficients for the distributed sextupoles. In the case of off-momentum particles, the equivalent quadrupole effect of the sextupoles is also introduced.

Modified lattice β-functions are first evaluated. The mismatch of the β-profile increases as a function of the space charge, particularly for positive values of the off-momentum parameter, $\Delta p/p$. Hamiltonian coefficients are computed from the perturbed β-values for given sextupole distributions, $\Delta p/p$ values and space charge fields.

Third-order Hamiltonian coefficients are generated from functions such as those introduced in section 3. Typical widths of third-order sum resonances in EPIC are given in Figure 1 as a function of space charge for a particular sextupole distribution and for a horizontal to vertical beam aspect ratio of 10:1. The assumed beam emittances correspond to 10 standard deviations of the expected Gaussian distributions at 14 GeV. The space-charge parameter, ΔQ, is chosen to be equal in the x and y directions, with ΔQ_h defined:

$$\Delta Q_h = Nr_e\beta_h^{**}/2\pi\gamma\sigma_h(\sigma_h+\sigma_v)$$

Fourth-order terms are computed from a large number of the third-order coefficients. The infinite sums, described in section 4, have each been approximated by a series of 15 leading terms. Subsequently, M H R Donald[6] has obtained solutions for the infinite series. In Figure 2, resonance-width contributions due to the sextupoles are plotted for the $2Q_v+2Q_h = 76$ resonance in EPIC. Also shown as a function of ΔQ are the Hamiltonian coefficients for the coupling resonance $2Q_v-2Q_h = 0$ (suitably scaled by $8R\varepsilon_h\ (1+\varepsilon_v/\varepsilon_h)$), and the variations, δQ_v, in the vertical tune (lowest two curves). Non-linear space-charge contributions to the resonances, and to δQ_v, have not been included in the curves.

Results indicate that the non-linear effects of the sextupoles are enhanced as ΔQ increases. Beam-beam limits will not depend on the non-linearities of the space charge fields alone, but on the combined effects of the sextupoles and the space charge forces. For the particular example chosen, there is significant horizontal-vertical coupling and change in vertical tune due to the sextupoles. With $\delta Q_v = -0.1$, the vertical tune decreases towards the $Q_v = 19$ resonance.

It is planned to continue studies with other sextupole distributions and Q values, and to compare the results with detailed trackings. Alternative tunes, close to even-integer values, are of interest because of the large decrease of β^{**} with ΔQ.

Lattices with superperiod number 2 have not been studied, but the present studies suggest they will have undesirable resonance properties.

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