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RESONANCE EXCITATION BY DISTORTION OF THE β-FUNCTION COUPLED WITH A SINGLE BEAM SPACE CHARGE FORCE*

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Summary

It is shown how a single beam space charge force can combine with a lattice design having large β variations, such as occurs in machines with low β insertions, to induce resonance behavior. Although the single beam space charge force may be highly nonlinear, it will, by itself, excite no resonances, since its azimuthal Fourier decomposition is essentially composed of 0th harmonic. However, low periodicity, large β variations are harmonically rich and can provide the necessary azimuthal harmonics for resonance excitation. The resonance characteristics of this type of system are developed. A strength parameter involving the linear tune shift and the β_{max} value is introduced. In particular, two cases are discussed: (1) where the force arises from the beam self-field and (2) where the force is induced by images in the surrounding boundaries. A comparison is made with the beam-beam force in terms of both the strength parameters (related to the linear tune shifts), and the nonlinear resonance behavior.

1. Introduction

Particle behavior in accelerators or storage rings can be characterized by betatron oscillations about a fixed equilibrium orbit.[1] Under certain circumstances, particles will exhibit resonance behavior or a growth in these betatron oscillations, which are, in general, induced by azimuthal harmonics of specific perturbing field components[2] (i.e. derivatives of these perturbing fields with respect to a transverse dimension, horizontal or vertical). Thus, if the perturbation has a (p-1)th derivative on the equilibrium orbit and this component has an nth azimuthal harmonic, then for the betatron tune near $\nu = n/p$, a resonance is excited with strength proportional to the nth harmonic of the perturbing field. This is an incomplete description in that particle motion in the vicinity of the resonance tune is for the case of nonlinear resonances with $p \geq 4$ significantly affected by nonlinear detuning arising from 0th azimuthal harmonics of all even ordered field components (i.e. even p), the lowest and in many cases the most substantial component being the octupole term, corresponding to p = 4. Although this nonlinear detuning may dampen a potentially explosive resonance, it cannot altogether suppress the growth characteristics. Particles whose tune is amplitude dependent can still "lock-into" a resonance and be drawn to large betatron amplitudes, although the time scale and amplitude growth are rather different from the case of explosive growth.[3]

There is a second aspect, in which the excitation of a resonance deviates from the simple picture of excitation through an nth azimuthal harmonic of a field component. A strict analysis demonstrates that, rather than the nth harmonic of a field component, it is the nth harmonic of a field component weighted with some power of the betatron amplitude function [i.e. $\beta^{p/2}(\theta)$].[2] In physical terms, it is clear that for a given perturbing nonlinear field, if particles are constrained to move at larger amplitudes, which results if the β-function is larger, then the resonance characteristics will be altered. However, in accelerators with a high periodicity, the β-function is composed primarily of a 0th azimuthal harmonic, the next contributing harmonic being related to the number of β-function oscillations per revolution. Thus, if a machine has 60 β-function periods, as in the Brookhaven AGS, then a Fourier decomposition produces the azimuthal components 0, 60, 120 .... With such a large separation between harmonics, it is not difficult to design a machine so that the higher harmonics of the β-function play no role in resonance excitation. The simple picture therefore holds. However, in machines designed with low symmetry,[4] as in storage rings or high energy accelerators, where a small number of specialized insertions are included in the lattice structure, this situation does not necessarily prevail and β-function variations must be dealt with.

There are 3 general features of the β-function influence on resonance excitation: (1) the β-function periodicity or lattice symmetry; (2) the β-function variation in magnitude, determining both the size of harmonic contribution and the richness in the harmonic content (i.e. the number of contributing harmonics); and (3) the extent to which the β-function harmonics and perturbing field harmonics are orthogonal, for it is the harmonic content of the appropriate product of β-function and perturbing field which actually induces the resonance.

All three of these features enter in a rather strong way in the specific case of low periodicity, high current storage rings with low β insertions.[4] In particular, the high current provides large nonlinearities of many orders arising from space charge fields; the large β-function required to obtain low β crossings provides the richness in azimuthal harmonic content; while the low periodicity makes the resonance tunes difficult to avoid.

Now, it is important to emphasize that we are considering here the space charge fields that arise from the self and image-fields of the beam itself. In other words, it is primarily a single beam phenomenon in the sense that the interaction of the two colliding beams does not produce the resonance excitation. Note the significant distinction. The beam-beam interaction is rich in azimuthal harmonics and rich in nonlinear field components. One does not need the β-function variation to provide the azimuthal harmonics. The beam-beam interaction, occurring essentially at one point in the azimuth provides this itself. In general, the single beam space charge forces are dominated by the 0th azimuthal harmonic. Thus, to induce nonlinear resonance behavior, it is the β-function variation which is required to produce the azimuthal harmonic content. It is this that is the basic substance of our model of resonance excitation: large β variations of low periodicity coupled with intensity induced single beam space charge fields.

We will apply our theory to the case of a periodicity-one lattice. The extension to higher symmetry is straightforward. Another simplification is the assumption of only a single excitation region in the lattice. In this regard, we may note that symmetric low β insertions[4] have two identical high β regions separated by a betatron phase of π. This implies that for even-ordered resonances, the strengths simply add--i.e. it is as if we had a single high β region of twice the strength. Note that for a perfectly centered beam in the high β region, only even-ordered resonances are ex-

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cited by the space charge force.

Odd-ordered resonances can be excited if the beams are not centered in the vacuum chamber at the high β regions. The excitation strength is then related to the precision of quadrupole placement relative to the equilibrium orbit in the high β regions. We will not consider such error-related excitation terms, restricting ourselves to the ideal situation of a centered beam exciting only even-ordered resonances. Similarly, we will not treat resonance excitation by errors in the quadrupole fields at the high β regions, the strength of these resonances imposing limits on the allowable field errors in the quadrupoles.

There are two classes of nonlinear forces elicited by the space charge of the beam; namely, the force caused by the beam self-field and that induced by the images formed in the surrounding walls. The important distinction between the two is that the former is a strong function of beam size, while the image field is essentially independent of the beam distribution.[5]

Consider the impact of the self-field on resonance excitation. In a manner similar to the beam-beam resonance analysis,[6] it can be shown that the quantity determining the resonance characteristics in this case is also proportional to the β-function and inversely proportional to the second power of the relevant size dimensions. In the beam-beam case it is found that the strength is proportional to β (interaction point)/(beam area), and for round beams is thus independent of β. However, the azimuthal harmonics of the force necessary for resonance excitation is still present for the beam-beam force since it is the beam-beam space charge force itself that is the source. Using a simple extension of the methods used in analyzing beam-beam resonances,[6] we find that in our case (i.e. the beam is large in the dimension where the resonance is potentially excited), the quantity determining the resonance excitation is proportional to $\beta/(\text{beam size})^2$. This is independent of β, which means that the azimuthal variation of the structure function, $\beta(s)$, is exactly cancelled by the azimuthal variation of the self-field space charge force. But there is no other azimuthal variation. The effective force, after incorporating the structure function of the lattice, is azimuthally constant. Thus, no resonance excitation can result. Note that we have neglected local size variation due to energy dispersion.

On the other hand, the space charge image field, being essentially independent of the beam charge distribution, has no azimuthal variation. It can therefore be anticipated that azimuthal harmonics introduced by the variation of the structure function can induce resonance excitation. In Section 2, we develop the resonance characteristics of such a system. In Section 3, we compare this type of resonance with beam-beam resonances.

2. Resonance Excitation By Single Beam Image Force

Space Charge Image Force

To be specific, we consider a parallel plate geometry. We restrict ourselves to an infinitely conducting metallic boundary, causing an image component of the space charge electric field. We do not include image contributions to the magnetic field, although we admit that they are not a priori negligible. It is presumed that such magnetic forces will not significantly affect the conclusions arising out of our resonance model. For beams not close to the boundary, the image fields for the assumed parallel plate geometry are somewhat insensitive to the transverse density distribution of the beam. We can therefore approximate the beam by an "infinitesimal wire". This simplification would not be possible with a circular geometry where the image field would vanish for an "infinitesimal wire" at the center. In this case, the image field is only non-zero for a beam displaced from the center (even if it is infinitesimal in extent) or for a centered beam with finite size. In the latter instance, the image field is, of course, sensitive to the transverse density distribution. To elucidate the principles implicit in the model proposed here, we will consider the simplified example of a symmetrically placed beam (with respect to the image boundary) of infinitesimal transverse size. In this limiting case, the circular geometry leads to no effect and we are left with the parallel plate geometry.

It can be shown[6] that for an infinitesimal wire beam symmetrically placed between two infinitely conducting parallel plates placed in the horizontal-longitudinal plane a distance 2h apart vertically, the vertical force at the horizontal position of the beam and within the plates, written as F_y, is given by,

$$F_y = \left(\frac{e\lambda}{4h\varepsilon_o}\right)\left(\frac{1}{\sin(\pi y/2h)} - \frac{2h}{\pi y}\right) \quad , \qquad (2.1$$

where
- y is the vertical coordinate with respect to the beam position at the center of the two plates,
- h is the half-distance between the plates,
- $\lambda = eN/C$ is the average linear charge density along the beam axis,
- N is the total number of particles in the beam,
- $C = 2\pi R$ is the ring circumference, R is the average radius,

and ε_o is the free space dielectric constant.

One Dimensional Equation of Motion

To obtain the one dimensional vertical equation of motion for a particle in the presence of the image force, we simply include that force, given in (2.1), in the equation for vertical betatron motion characterized by the lattice structure for the ring. Thus, we have[1]

$$y'' + K(s)y = F_y/m\gamma c^2 \quad , \qquad (2.2)$$

where
- $K(s)$ is the gradient forcing function for the lattice,
- m is the particle rest mass,
- γ is the total energy of the particle in units of its rest mass, we have taken the particle velocity to be close to c, the velocity of light,
- s is the distance measured along the lattice equilibrium orbit from some reference position,

and y is the vertical particle displacement from the equilibrium orbit.

To describe the unperturbed motion, we introduce[1] an "amplitude function", $\beta(s)$. Introducing a tune, ν, a phase for the independent variable, $\theta(s)=\int_0^s d\rho/(\nu\beta(\rho))$, and a new displacement variable $t=y/hu^{\frac{1}{2}}$, where we have defined $u(s) = \beta(s)/\beta_{av}$, with $\beta_{av} = R/\nu$, then Eq. (2.2), using (2.1) for the force, F_y, transforms to

$$\ddot{t} + \nu^2 t = -2\nu\Delta\nu_{IM}\, u^2 t H(u^{\frac{1}{2}}t) \quad . \qquad (2.3)$$

Here, $\Delta\nu_{IM}$ is just the tune shift caused by the image space charge force,[7]

$$\Delta\nu_{IM} = -\pi N r_o R/(48\gamma\nu h^2) \quad , \qquad (2.4)$$

where r_o is the classical radius of the particle ($=e^2/4\pi\varepsilon_o mc^2$). With $z = \pi u^{\frac{1}{2}}t/2$, $H=(6/z)[1/\sin z-1/z]$. Differentiation is with respect to the betatron phase angle θ, which is similar but not identical to the azimuth, θ_z; and ω is considered as a function of $\theta(s)$.

Fourier Decomposition of $\beta(\theta)$

In an alternating gradient structure with many cells, the amplitude function modulates with a high periodicity. The harmonic structure is therefore widely spaced. In looking for resonance effects, this rapid modulation is of little consequence and the β-function in these parts of the azimuth can be replaced by its average value, β_{av}. In the insertions, however, the variation of $\beta(\theta)$ is of a much larger magnitude. In particular, in low β insertions for storage rings,[4] β may reach values many times larger than β_{av}. However, these regions where β rises to large values occur in only short azimuthal extents. We therefore can approximate this effect with a δ-function in azimuth.

We will require various powers of the β-function. With the above discussion in mind, we write for the mth power of $\omega(\theta) = \beta(\theta)/\beta_{av}$, in the case of M identical large fluctuations,

$$\omega^m(\theta) \simeq 1 + \frac{2\pi\Gamma_m}{M} \sum_{i=1}^{M} \delta(\theta-\theta_i) \quad . \tag{2.5}$$

We obtain Γ_m by integrating over θ, resulting in

$$\Gamma_m \simeq \frac{2M}{C} \int_0^{\Delta/2} [\omega(s)]^{m-1}\, ds \quad , \tag{2.6}$$

where Δ is the total length over which the β-function modulation extends: $\Delta/C \ll 1$.

Notice that the power of ω is reduced by one. This arises from the fact that in terms of the betatron phase, which is the relevant "time" variable, the effective "length" over which the β-function rises and falls shrinks. This contraction in the effective azimuth results in a diminution of the factor Γ_m, i.e. the strength of resonance excitation, by one power of $\omega_{max} = \beta_{max}/\beta_{av}$. In fact, it is clear that for $\omega_{max} \gg 1$, we will have an estimate for Γ_m of the order of $\Gamma_m \sim M\Delta(\omega_{max})^{m-1}/mC$, roughly independent of the details of β-function shape. For $\omega(s)$ linear in s, Γ_m approaches precisely this value, while for a quadratic dependence on s, we have $\Gamma_m \simeq (M\Delta/C)[\omega_{max}^{m-1}/(2m-1)]$. We will use this latter value as an approximation to the actual value of Γ_m, which can be obtained from the complete expression, Eq. (2.6), for any given s dependence of $\omega(s)$. A Fourier expansion of (2.5) in terms of the "effective azimuthal variable", θ, results in

$$\omega^m(\theta) = 1 + \Gamma_m + 2\Gamma_m \sum_{\ell=1}^{\infty} \cos \ell M\theta \quad , \tag{2.7}$$

where we have chosen the coordinate zero such that $\theta_i = 2\pi(i-1)/M$. For $\Gamma_m \gg 1$, (2.7) can be written $\omega^m(\theta) \doteq \Gamma_m u(\theta)$, with $u(\theta)$ defined through (2.7).

Resonance Equations and Invariant

To obtain the equations of motion under resonance conditions as well as the resonance invariant, we transform to amplitude and phase variables, I and ϕ respectively, related to t and $\dot{t}$ by $t = \sqrt{I}\cos\phi$, $\dot{t} = -\nu\sqrt{I}\sin\phi$. The resulting equations for I and ϕ are $\dot{I} = 2\dot{t}(\ddot{t} + \nu^2 t)/\nu^2$, $\dot{\phi} = \nu - \cos\phi(\ddot{t} + \nu^2 t)/\nu\sqrt{I}$; or,

$$\begin{aligned} \dot{I} &= 2I\, \Delta\nu_{IM}\, \omega^2 \sin 2\phi\; H(\omega^{\frac{1}{2}} t) \; , \\ \dot{\phi} &= \nu + 2\, \Delta\nu_{IM}\, \omega^2 \cos^2\phi\; H(\omega^{\frac{1}{2}} t) \; . \end{aligned} \tag{2.8}$$

It can be shown that an approximate expansion for H in powers of z yields,

$$H = 1 + \frac{12}{\pi^2} \sum_{n=2}^{\infty} \omega^{n-1} \left(\frac{I}{4}\right)^{n-1} \cos^{2n-2}\phi \; . \tag{2.9}$$

Defining a new strength parameter $\Gamma_{IM} = \Delta\nu_{IM}\Gamma_2$, where $\Gamma_2 = (M\Delta/3C)(\beta_{max}/\beta_{av})$, and introducing a new amplitude variable, $\alpha = \omega_{max} I/4$, we can obtain phase and amplitude equations from (2.8). The detuning term is obtained by averaging over phase and azimuth, while the resonant excitation term is obtained by neglecting rapidly oscillating terms. If we introduce the slowly varying phase variable $\Psi = \phi - (\ell M/p)\theta$, where p and ℓ are integers such that $\delta = \nu + \Gamma_{IM} - \ell M/p$ is small, than we can write the resonance equations,

$$\begin{aligned} \dot{\Psi} &= \delta + \Gamma_{IM}\,[F(\alpha) + \cos p\Psi\; V_p(\alpha)] \; , \\ \text{and} \qquad \dot{\alpha} &= \Gamma_{IM}\, p \sin p\Psi\; V_a(\alpha) \quad , \end{aligned} \tag{2.10}$$

where the detuning and resonance functions are

$$F(\alpha) = \frac{72}{\pi^2} \sum_{n=2}^{\infty} \frac{\alpha^{n-1}}{2n+1} \frac{(2n-1)!!}{(2n)!!} \quad , \tag{2.11}$$

$$V_p(\alpha) = \frac{72}{\pi^2} \sum_{n=p/2}^{\infty} \frac{\alpha^{n-1}}{(2n+1)} \frac{1}{2^{2n-1}}\, {}^{2n}C_{n-p/2} \; , \tag{2.12}$$

and

$$V_a(\alpha) = \frac{72}{\pi^2} \sum_{n=p/2}^{\infty} \frac{\alpha^{n}}{n(2n+1)} \frac{1}{2^{2n-1}}\, {}^{2n}C_{n-p/2} \; . \tag{2.13}$$

${}^rC_q$ is the usual binomial coefficient. Note that $V_a'(\alpha) = V_p(\alpha)$, as required for the existence of a resonance invariant. We can relate the variable α to the standard emittance parameter (Area = π X emittance): $\alpha = \beta_{max}\, \epsilon/4h^2$. Defining γ by $\gamma = \epsilon/\epsilon_{rms}$, where ϵ_{rms} is the beam emittance, we have that α and γ are connected by

$$\alpha = \tau\gamma = \beta_{max}\epsilon_{rms}\gamma/4h^2 \quad . \tag{2.14}$$

In graphs depicting the various resonance and detuning functions, we use as independent variable, σ, the rms displacement in units of rms beam size, related to γ by $\sigma = \sqrt{2\gamma}$. We have assumed previously that $\Gamma_2 \gg 1$. We can extend our analysis to include the case $\Gamma_2 \lesssim 1$ by a simple modification of (2.10). In a manner similar to the analysis for the case $\Gamma_2 > 1$, we obtain in general,

$$\dot{\Psi} = \delta + (\Gamma_{IM} + \Delta\nu_{IM})F(\alpha) + \Gamma_{IM} V_p(\alpha) \cos p\Psi \; . \tag{2.15}$$

3. Comparison With Beam-Beam Resonance

Because of the large detuning characteristic of space charge forces, and this includes the image force, the nonlinear resonances excited by these forces influence particle motion primarily by the lock-in process.[8] The resonance behavior of such systems is characterized by three amplitude functions,[8] (1) the detuning function, (2) the adiabatic boundary function, and (3) the instantaneous trapping function. The detuning function is related to $\bar{\delta}$ the distance of the tune from the resonant tune ($\nu_{RES} \sim n/p$):

$$\bar{\delta} = d(\sigma) = [\xi D_{bb}(\sigma), 2(\Gamma_{IM} + \Delta\nu_{IM})D(\sigma)] \quad , \tag{3.1}$$

where ξ is the beam-beam strength parameter (roughly the linear tune shift), the factor of 2 corresponds to there being two high β lengths for every one beam crossing, $D_{bb}(\sigma)$ is the detuning for the beam-beam interaction and is given in Ref. (8), $D(\sigma) = -(F(\sigma) + 1)$. Equation (3.1) defines the amplitude of the lock-in islands for a given tune. The adiabatic boundary function determines whether or not particles will be trapped. The trapping criterion can be written as an upper limit on the speed of tune variation,

$$\delta\upsilon_{rev} \le r(\sigma) = [\xi^2 R_p^{bb}(\sigma),\ 4\Gamma_{IM}(\Gamma_{IM} + \Delta\upsilon_{IM})R_p(\sigma)] \quad (3.2)$$

where $R_p^{bb}(\sigma)$ is the adiabatic boundary function for the beam-beam interaction, given in Ref. (8), and $R_p(\sigma)$ is the corresponding function for the image field effect, $R_p(\sigma) = |2\pi p V_p(\sigma) F'(\sigma)|$. The function F' is derived from F by differentiation with respect to α. The instantaneous trapping function is just the total number of particles engulfed by the islands at a given σ as the islands pass through the beam. It gives the number of particles that potentially can be trapped at a given instant. If we approximate the instantaneous islands by a ring in betatron phase space, we have for the instantaneous trapping function at some amplitude σ,

$$P_T = e^{-\gamma_-} - e^{-\gamma_+} \quad , \quad (3.3)$$

where $\gamma_\pm$ are the upper and lower bounds of the ring,[8] and functions of σ: $\gamma_\pm = \gamma \pm (2/\tau)|V_p/F|^{\frac{1}{2}}[1+\Delta\upsilon_{IM}/\Gamma_{IM}]^{-\frac{1}{2}}$. We obtain P_T for the beam-beam interaction from Ref. (8).

We compare the functions $d(\sigma)$, $r(\sigma)$ and $P_T(\sigma)$ for the beam-beam and image resonances in Figs. 1, 2, and 3 respectively, taking as an example, the 6th order resonance (p=6). For the strengths, we use typical high energy storage ring parameters. Taking R = 1000 m, $\upsilon = 40$, h = 3 cm, $\gamma = 400$, and an average current I = 10 A, we have from (2.4), $\Delta\upsilon_{IM} = 9.2 \times 10^{-3}$. For the lattice parameters, $\beta_{max} = 1000$ m, $\beta_{av} = R/\upsilon = 25$ m, M = 1, and $\Delta/C = 0.016$, and therefore, $\Gamma_{IM} = 1.94 \times 10^{-3}$. Also, taking a beam emittance, $\epsilon_{rms}\ \gamma = 6.0 \times 10^{-6}$ rad-m, we have $\tau = 4.17 \times 10^{-3}$. For purposes of comparison, we take a beam-beam strength, $\xi = 1 \times 10^{-3}$. We also show the impact of decreasing the beam energy to γ=100. Note that the quantities $\Delta\upsilon_{IM}$, Γ_{IM} and τ are all inversely proportional to γ and so increase by a factor of four with this energy change.

From the curves we can see the following: (1) The detuning function for an image-β resonance is a much weaker function of amplitude than for the beam-beam resonance. This means for a given tune shift from the resonant tune, the trapping islands will move much further in amplitude in the image-β resonance case. (2) The amplitude boundary function for the image-β resonance is very sensitive to energy and does not level off with amplitude up to σ=5. The meaning of these curves is that tune rates (per revolution) slower than $r(\sigma)$ allows lock-in or particle trapping at σ. Thus, compared to beam-beam resonances, trapping for image-β resonances is less likely. (3) For image-β resonances, the fraction of particles that can be instantaneously trapped, P_T, is smaller by an order of magnitude than for beam-beam resonances.

Note that in the lower energy case (γ=100), the filling of the aperture is ~ 50% at the β_{max} location. In fact, it is clear that an increase in the chamber radius, h, at this location will decrease the strength of the image-β resonance, since $\Delta\upsilon_{IM}$, Γ_{IM}, and τ all vary as $1/h^2$. However, it should be kept in mind that it is precisely at this location that we must have quadrupoles to focus the rising β function. In order to achieve the required field gradients, to accommodate high energy particles, and to accomplish this while constrained to a maximum quadrupole pole tip field, we are driven in the direction of smaller, not larger, chamber dimensions.

4. Conclusions

We have considered the effect of large, low periodicity β fluctuations. When coupled with the space charge fields of an intense beam, we have concluded that resonances can be excited by the image field but not by the beam self-field. The comparison of one-dimensional image-β resonances with beam-beam resonances suggests that they should not be ignored in the design of storage rings.

References

1. E.D. Courant and H.S. Snyder, Ann. Phys. 3, 1 (1958)
2. See, e.g. A. Schoch, CERN Rept., CERN 57-23 (1958).
3. See, e.g. A.W. Chao and M. Month, Nucl. Instrum. Methods 121, 129 (1974).
4. See, e.g. "ISABELLE", BNL 18891, May, 1974.
5. M. Month and R.L. Gluckstern, Part. Accel. 6, 19(1974
6. See, e.g. M. Month, BNL Rept., CRISP 75-1 (1975).
7. L.J. Laslett, Summer Study, BNL 7534, p.324 (1963).
8. M. Month, "Nature of the Beam-Beam Limit in Storage Rings", this conference.

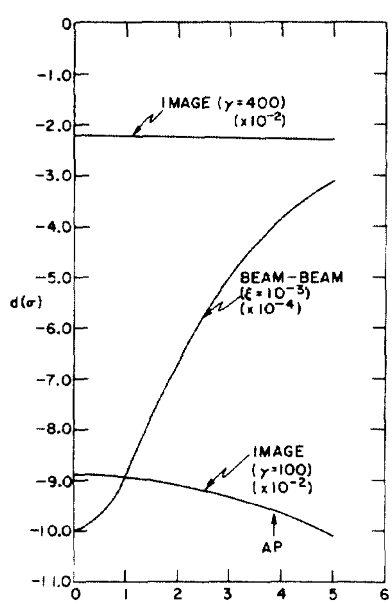

Fig. 1. Detuning function, $d(\sigma)$. AP is the position of the physical aperture boundary at the β_{max} location in the case γ = 100.

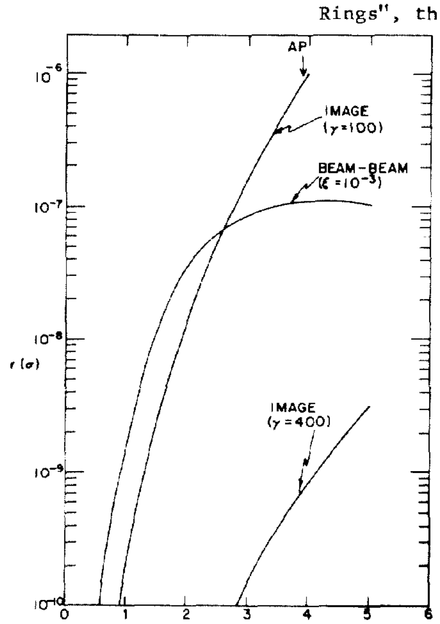

Fig. 2. Adiabaticity boundary function, $r(\sigma)$. 6th order resonance.

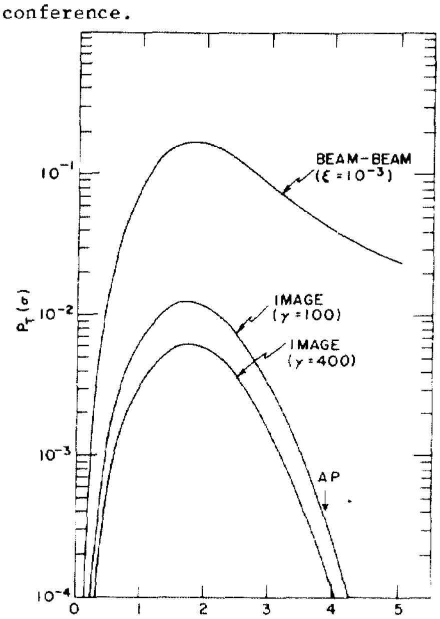

Fig. 3. Trapping probability function, $P_T(\sigma)$. 6th order resonance.