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NATURE OF THE BEAM-BEAM LIMIT IN STORAGE RINGS*

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Summary

The experimental results relating to the beam-beam interaction in electron storage rings and the ISR (p-p collisions) are considered. The question of whether or not stochasticity is implied by these results is discussed. It is argued that all the available evidence on the beam-beam limit is not inconsistent within an isolated resonance framework. A model which qualitatively fits the observations, one which is derived from classical resonance theory, is proposed.

1. Introduction

Observations on the effects of the electromagnetic interaction between two colliding beams have been made for electrons and positrons colliding head-on[1-3] and for coasting proton beams[4-7] colliding at a large angle. It is found that the primary parameters determining the beam lifetime are the strength of the interaction and the operating tunes of the beams. There is thus a qualitative similarity between e^+e^- collisions and p-p collisions. However, in e^+e^- collisions, we are dealing with beam-beam strengths almost two orders of magnitude larger than for p-p collisions and lifetimes at least an order of magnitude less. Nonetheless, the electromagnetic interaction is essentially the same and we expect the influence to arise from the same source.

Although it is apparent that the nonlinear resonance excitation characteristic of the interaction of two beams plays a dominant role, the precise mechanism through which the nonlinear beam-beam force exerts its influence on the beam lifetime is a matter of some dispute. The central point of the dispute is whether the nonlinear beam-beam resonances act in a manner consistent with a conventional, isolated resonance treatment[8,9] or whether the observations are a result of the combined influence of many resonances acting simultaneously.[10]

The conceptual basis for the multi-resonance approach is the simulation of stochastic behavior by the interaction of overlapping resonances. Although there exists a preliminary attempt at developing a theory of such stochastic phenomena, starting with a conservative Hamiltonian, the primary means of study has been through numerical experiments.[11] The results of such experiments are inconclusive. On the one hand, it appears clear that for sufficiently large beam-beam strengths, the system is unstable. On the other hand, the dependence of the instability on various parameters such as tune, time and initial conditions is vague and uncertain. The situation is further compounded by the numerical accuracy problems inherently associated with strongly nonlinear equations.[12] In fact, the only conclusions that can be derived from this approach are rough limiting strengths.

However, the existence of a limiting strength is not in question. The question is whether or not the observations are in any way connected with this "stochastic limit". It is our contention here that the observations in both e^+e^- collisions and p-p collisions are not related to the presence of a stochastic limit to the beam-beam strength, but on the contrary can be described in terms of an isolated resonance framework. Such a description does not contradict the conclusions arrived at in numerical experiments, but rather supercedes them.

As we have already pointed out, e^+e^- and p-p collisions as they exist have greatly different strengths as well as beam lifetimes. A satisfactory theory must however be able to unify the observations of these two diverse systems into a common base. We propose here that the common base is simply the process of resonant lock-in.[10,13]

Thus, we have the following picture: There is a particle distribution in betatron amplitude. The beam beam strength parameter and the tune determine the amplitude of lock-in islands. For beam-beam resonances, these produce only small amplitude modulation if the external parameters (tune and beam-beam strength) are fixed in time. However, as these parameters change, particles can be trapped in the stable islands and be transported to larger amplitudes. Thus, a small fraction of particles can be lost. Continued loss is caused by a resonance feeding process, with trapping and transport to the physical aperture repeating.[15-19]

The element common to electron collisions and proton collisions is the resonant lock-in process. The factors which give them their distinctly different behavior are (1) the mechanism for time variation of external parameters and (2) the resonance feeding mechanism. For electron collisions, it is the synchrotron motion that induces a time variation of tune and beam-beam strength, while resonance feeding is a result of quantum fluctuations.[20] For proton collisions, the beams are coasting and the beam-beam strength is fixed in time. Both resonance crossing (trapping and transport) and resonance feeding are a consequence of the fluctuations of tune arising from intra beam scattering via the chromaticity.[16,17,19]

In Section 2, we review the trapping theory for beam-beam resonances for head-on collisions.[15,16] We presume that such a resonance model can be applied to p-p collisions at large crossing angle by appropriately modifying the strength parameter. We emphasize the distinctive amplitude dependence of both the nonlinear detuning and the resonant widths.[14-16,18] We discuss the nature of both odd and even ordered resonances. We show how lock-in and particle transport can occur as a result of a time variation of tune or beam-beam strength.

Using the resonance lock-in mechanism, we construct models for beam growth and loss in electron collisions in Section 3 and in proton collisions in Section 4. We find our models in qualitative agreement with the experimental observations.

2. Beam-Beam Resonances

Beam-beam resonances differ considerably from the normal multipole resonances induced by magnetic fields outside the beam aperture. The two most striking differences are: (1) The nonlinear detuning remains much larger than the resonant widths even for very large amplitudes. This is in marked contrast to the case of conventional resonances, where we expect the resonant width function to dominate at physically relevant am-

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plitudes (i.e. within the physical aperture). This dominance threshold denotes the start of an unstable region of betatron amplitude. Such an unstable region is not of relevance for the beam-beam interaction. (2) The resonant widths induced by the beam-beam interaction have a rather slow amplitude variation (increasing linearly at large amplitudes for all resonances),[18] again contrasting with the behavior of multipole resonances, where they rise with increasing amplitude with a power corresponding to the order of the nonlinear resonances.[8]

Although the details of the topology in betatron phase space is quite different, the basic lock-in process remains. Contrary to multipole resonances, it is the only way in which isolated beam-beam resonances can influence the particle motion. "Fast crossing" effects[8] can in principle occur, but in general the large detuning of the beam-beam interaction makes lock-in a dominant feature.

Trapping Amplitude

Since the detuning is much larger than the resonant width function, the island amplitude is determined essentially independent of the latter. It is found that the amplitude of the trapping island, σ, is given by the solution of the equation,[15]

$$D(\sigma) = \frac{\bar{\nu}}{M\xi} \quad , \tag{2.1}$$

where $\bar{\nu}$ is the distance of the tune from the resonance,
ξ is the beam-beam strength parameter,
M is the number of identical collisions,
σ is the betatron phase space amplitude in units of the rms beam amplitude,
and $D(\sigma)$ is a function related to the nonlinear detuning, plotted in Fig. 1 for both a ribbon beam and a round beam.

Rate Criterion

Particles in betatron phase space are influenced by the resonance only when their amplitudes are in the vicinity of the trapping islands. For beam-beam resonances, the islands extend instantaneously only over a small amplitude range, a reflection of the large detuning relative to the resonant width. Thus, from the point of view of particle stability, the static phase space topology appears innocuous, producing only a weak amplitude modulation. If, however, the external parameters $\bar{\nu}$ and ξ are time dependent, then the trapping amplitude is changing with time. This means that particles can lock-in to the resonance (be trapped in the passing islands) and be transported to larger amplitudes. But, in order for trapping to result, the changing particle amplitude due to the resonance must be at a rate at least as fast as the speed of the passing islands.[8,15,16] The question is, how slowly do the external parameters, $\bar{\nu}$ and ξ, have to be changing in order for trapping to occur? The answer is in the form of a trapping criterion, stated as an upper limit on the rate of variation of the external parameters.

For M identical collision points, and writing the resonant tune

$$\nu_{RES} = m/p \quad , \tag{2.2}$$

we have that the azimuthal harmonic, m, exciting the resonance, must be a multiple of M. For ideal collisions, i.e. beam centers coinciding, then the order of the exciting resonance, p, must be even. In this case, the trapping criterion is[15,16]

$$\frac{1}{M^2\xi}\,\delta_{rev}\left(\frac{\bar{\nu}}{\xi}\right) \le R_p(\sigma) \quad , \tag{2.3}$$

where δ_{rev} means the rate of change per revolution,
and $R_p(\sigma)$ is a function related to the resonant width function and the nonlinear detuning, and is plotted for a few even resonances in Fig. 2(a) for a round beam and in Fig. 2(b) for a ribbon beam.

The excitation of odd-ordered resonances requires nonideal collisions. In this case, the trapping criterion can be shown to be[15,16]

$$\frac{1}{M^2\xi\Gamma}\,\delta_{rev}\left(\frac{\bar{\nu}}{\xi}\right) \le R_p(\sigma) \quad , \tag{2.4}$$

where the odd resonance excitation factor is given by

$$\Gamma = \frac{1}{\sqrt{2M}}\left|\sum_{\ell=0}^{M-1}\sigma_\ell\, e^{im\theta_\ell}\right| \quad , \tag{2.5}$$

with θ_ℓ the azimuth of the ℓ-th collision point,

and σ_ℓ is the displacement of the weak beam center from the strong beam center, in units of the rms beam amplitude.

The R functions for a few odd resonances are shown in Figs. (3a) and (3b). If the errors σ_ℓ are uncorrelated, then

$$\Gamma = \frac{1}{\sqrt{2M}}\langle\sigma\rangle_{rms} \quad , \tag{2.6}$$

where

$$\langle\sigma\rangle^2_{rms} = \frac{1}{M}\sum_{\ell=0}^{M-1}\sigma_\ell^2 \quad . \tag{2.7}$$

Trapping Process

Each particle in a beam can be identified by giving $\bar{\nu}$, ξ and their rates of change $\delta_{rev}\,\bar{\nu}$ and $\delta_{rev}\,\xi$. These will in general be functions of time. From (2.1) we can determine the amplitude path of each particle *if* it locks-into the resonance. With (2.3) or (2.4), we can determine whether in fact the particle will be trapped around the stable island center. Thus, by solving (2.1) for σ at various instants, we can evaluate the ℓ.h.s. of either (2.3) or (2.4) as a function of σ and superimpose such particle paths on the curves for the corresponding R_p function. Interpreting the R curve as a boundary, we can see that (1) a particle can be captured by a passing island, or it can be bypassed, (2) a particle can be dragged over a large amplitude range; and (3) a particle can be dropped from an island which has become leaky.

We have made the implicit assumption that a particle passing through resonance either will be trapped or it will remain essentially unaffected. This is not entirely correct for two reasons: In the first place, even if the islands pass by rapidly, there may be a residual effect. By presuming lock-in to be the dominant source of amplitude change, we are simply ignoring this small influence of "fast crossing". Secondly, there is obviously no sharp boundary for the island speed which can give us a well defined trap/no trap criterion. The boundary is undoubtedly fuzzy. This being the case, results and conclusions sensitive to the precise boundary location must be interpreted with this limitation in mind. However, it is our primary aim here to show qualitatively how the special features of beam-beam resonances manifest themselves in beam growth and beam loss. To achieve this end, our assumption of a sharp lock-in boundary seems reasonable.

3. Electron Collisions

We propose that the beam-beam interaction decreases the beam lifetime by introducing a resonance aperture within the physical aperture, thereby decreasing the quantum lifetime. The transport between the resonance and physical apertures is mediated by the time modulation of both the beam-beam strength parameter, ξ, and the tune, $\bar{\nu}$. These are induced by the much faster synchrotron oscillations and are, relative to the quantum fluctuation time scale, immediate.

Lifetime estimates are complicated by the fact that the resonance influences only a select group of particles within a six dimensional phase space in the three dimensions, horizontal and vertical transverse as well as longitudinal (synchrotron). Thus, the lifetime must reflect a feeding into an intersection of three spaces. Generally, the quantum lifetimes for the three spaces, when there are no resonance effects, are independent and can be determined separately.

Because of the complicated nature of the impact of the beam-beam resonances on the lifetime, we will be content with showing how particular particles in synchrotron and betatron phase space can have a resonance aperture well within the physical aperture. For this purpose, we consider only one dimensional resonances and ideal collisions. The latter implies only even-ordered resonances.

SPEAR: In SPEAR,[1] the dominant source of external parameter time modulation is the ξ modulation at twice the synchrotron frequency due to an effect caused by the low β^*.[21] In an interaction with low β^*, meaning essentially that $\beta^* \leq$ rms bunch length, then the strong beam strength parameter, ξ, depends on the azimuthal position of the weak beam particle relative to the synchronous particle. This is a result of the significant β change along the bunch length coupled with the changing density distribution as the weak beam particle passes through the strong beam. For an ideal collision, where the centers of the two bunches coincide at the minimum β, a particle at the azimuthal center of one bunch sees the maximum density of the other bunch at the minimum β, β^*. However, particles at azimuths different from the synchronous particle see the peak density at higher values of β, and so the force over the collision region is larger. Thus, ξ modulates as particles execute synchrotron oscillations. For a symmetric distribution around the synchronous particle, the frequency of the modulation is twice the synchrotron frequency.

A particle can now be designated by three parameters $\bar{\nu}$, ξ, and k, the particle's synchrotron amplitude (in units of the rms beam synchrotron amplitude). At each instant, certain particles in betatron phase space will be affected by the resonance. We will treat the case of the 8-th order resonance. Since SPEAR has two collisions per revolution (M=2), this occurs at 1/4 integer tune (i.e., $\nu_{RES} = 5.25$). The amplitude path can be traced by solving (2.1) for σ. Thus, the ℓ.h.s. of (2.3) can be treated as a function of σ, which we call $r(\sigma)$. In Fig. 4 we compare $r(\sigma)$ (for some representative values of $\bar{\nu}$, ξ and k) with $R_8(\sigma)$ (for a ribbon beam). Note the high sensitivity to the strength parameter, ξ. It is also evident that the rate of ξ variation is such that higher order resonances will play only a minor role unless ξ is substantially larger than $\xi \approx 0.03$. We might expect a beam-beam limit from the 8-th order resonance somewhere in this area. Beam loss is a consequence of r remaining below R as σ moves to the physical aperture from the initial trapping amplitude. Beam growth occurs if, after trapping, the R curve is crossed, allowing particles to leak out of the trapping island at a larger amplitude.

ACO: In ACO,[2] the dominant modulating source is the tune modulation due to the chromaticity. Using ACO parameters, we can plot curves analogous to those for SPEAR. These are given in Fig. 5. We see that for small ξ (e.g., $\xi = 0.015$), transport to the aperture for even the 6-th order resonance does not occur. However, beam size growth can occur, with substantial effect seen due to the 6-th and 8-th order resonances. For $\xi = 0.03$, however, beam loss can result from transport to the physical aperture caused by the 6-th, 8-th, and even the 10-th order resonances. Although it cannot be seen from Fig. 5, a more extensive set of such curves allows an estimate of stopbands, i.e. ranges of tune where beam loss occurs. At a strength of $\xi = 0.03$ we estimate a stopband for the 8-th order resonance, $-0.02 \lesssim \bar{\nu} \lesssim 0$, while the 10-th order stopband is somewhat smaller, $-0.01 \lesssim \bar{\nu} \lesssim -.005$.

Thus, a qualitative picture of ACO near the beam-beam limit emerges. At a certain ξ level, beam size growth results. Then, as ξ increases, the beam lifetime diminishes as resonance apertures appear within the physical aperture. As high order resonance stopbands appear and grow, the regions of operating tune shrink.

Note that because of the more rapid rate of variation of ξ in SPEAR as compared to the tune rate in ACO, for similar strengths, we expect the higher order resonances to play a more significant role in ACO. This is seen by comparing Fig. 4 for SPEAR with Fig. 5 for ACO.

4. Proton Collisions

For high symmetry collisions such as at the ISR,[4-7] where M=8, resonance excitation occurs by virtue of deviations from ideal collisions. The main source of excitation in p-p collisions is orbit misalignment at the interaction points. Thus, the dominant excitation is due to odd-ordered resonances.

It has become apparent that resonance feeding in coasting proton beams could be mediated by a particle scattering process in the intense ISR beams. In particular, intrabeam scattering, the dominant of such scattering processes, can induce momentum diffusion and through the chromaticity, tune drift. In this way, particles are fed into the resonant tune range.[16,17,19]

However, bringing the particles into resonance is not enough. In a coasting proton beam, where there is no periodic tune modulation, such as is caused by the synchrotron motion in a bunched beam, we might expect that in the presence of sufficient nonlinear detuning, the resonances would be quite harmless, producing only a small betatron amplitude modulation. This is the case for beam-beam resonances. Thus, it has been suggested that, in addition to feeding the resonance, the combination of tune diffusion and resonant streaming can produce amplitude growth sufficient to reach the physical boundary.[22]

By using a random walk model for the intrabeam scattering,[16] we can estimate the resulting tune drift and the rate of tune variation, these parameters governing the movement of trapping islands in the betatron phase space and determining whether or not a given particle will be trapped in an island. It is important to recognize that the process of tune drift is a random one and therefore each particle will have a complicated time dependence of its tune. However, since each momentum step is equivalent to a small quadrupole, since there are many steps per revolution, on the average tending to cancel, and further since the tune shift is by definition a vector sum of these quadrupoles over a revolution, then we can represent the tune shift, not by the quadrupole fluctuations within the revolution period, but rather by the sum, i.e. the tune drift. In other words, the distribution in tune drift after one revolution (many steps) is a representation of the tune speed distribution. As representative of this distribution, we can choose the rms value, $\delta\nu_{rev}$, simply related to the diffusion coefficient for tune diffusion, D_ν, by,

$$\delta\nu_{rev} = \sqrt{2D_\nu T_{rev}} \quad , \tag{4.1}$$

where T_{rev} is the revolution period. The meaning of this quantity is that we can expect instantaneously that 95% of the particles will have tune speeds not larger than twice this value. Any given particle will have a complex history of speeds. In fact one can see that this complexity provides a qualitative mechanism for the creation of halos in a freshly scraped beam. A given particle can be trapped at a small amplitude, while $\delta\nu_{rev}$ is small, be transported to a larger amplitude, and then dropped as $\delta\nu_{rev}$ gets larger and the island becomes leaky (i.e. when the trapping boundary is crossed).

The quantity $r(\sigma)$ from (2.4), for odd resonances, is treated here as a constant, and for uncorrelated errors can be written

$$r(\sigma) = r = \frac{\sqrt{2}\ \delta\nu_{rev}}{M^{3/2}\xi^2 \langle\sigma\rangle_{rms}} \quad . \tag{4.2}$$

With this simple form for r, we can attempt to estimate the loss rate to a physical aperture. We do this by finding the fraction of particles trapped and then solving the random walk problem in the presence of an absorbing barrier. The resulting loss rate is simply,[16]

$$\dot{N}/N = -\frac{P_T D_\nu}{\Delta\delta} \quad , \tag{4.3}$$

where Δ is the total tune spread in the beam (assuming a uniform distribution),

δ is the distance in tune to the aperture, and can be obtained from Fig. 1 (ribbon beam) as the tune distance corresponding to the distance from an initial trapping amplitude to the aperture amplitude,

and P_T is the trapping probability.

The instantaneous trapping probability is simply the number of particles instantaneously in the islands at some amplitude. For a Gaussian amplitude distribution, we plot in Fig. 6, $P_T(\sigma)$ for some odd resonances, assuming a strength given by $\langle\sigma\rangle_{rms} = 1$. For different strengths, we simply note that P_T is approximately proportional to $\langle\sigma\rangle_{rms}^{\frac{1}{2}}$. In any given loss rate computation, we can take some typical values of $P_T(\sigma)$ in the trapping region. To find the origin of the trapping region we use an expanded scale and replot Fig. 3(b) in Fig. 7 for small σ.

Using ISR parameters, we have $D_\nu = 5.54 \times 10^{-11}$/sec, $\delta\nu_{rev} = 1.87 \times 10^{-8}$, $\xi = 3.94 \times 10^{-4}$ per interaction, giving $r = 7.51 \times 10^{-3}$ for $\langle\sigma\rangle_{rms} = 1$. Taking $\Delta = 0.04$, and $\sigma_{AP} = 7$, we obtain, using Figs. 1 and 7, a loss rate $\dot{N}/N = -(7.9, 2.9, 1.1, \text{and } 0.1) \times 10^{-3}$/min for 5-th, 7-th, 9-th and 11-th order resonances, respectively.

It must be remembered that these loss rates are induced by particles in a selective betatron amplitude range depending on which resonance is excited. For example, the loss rate due to the 5-th order resonance involves particles with $\sigma \gtrsim 0.85$, i.e. near the beam center; while, for the 11-th order resonance, only particles with $\sigma \gtrsim 3.7$ are instantaneously affected. Thus, the higher order resonances are not as harmful because to sustain the rate they depend more heavily on the feeding mechanism to refill the amplitude region depleted by the resonance. The loss rate over a long time will therefore be determined primarily by the feeding process, a part of which is reflected in halo formation.

Because of the maximum in the resonance functions, see Fig. 3b, a sufficiently small excitation strength will remove the trapping potential from a resonance. The trapping region shrinks to zero, leading to a tolerance on $\langle\sigma\rangle_{rms}$. Taking the rms beam size to be, $\sigma^* = 1.62$ mm, the tolerances on the rms orbit alignment are found to be 0.05 mm for the 5-th order resonance and 0.13 mm for the 7-th.

5. Conclusions

The isolated resonance theory presented here gives results qualitatively in agreement with observations at both electron rings and the ISR.

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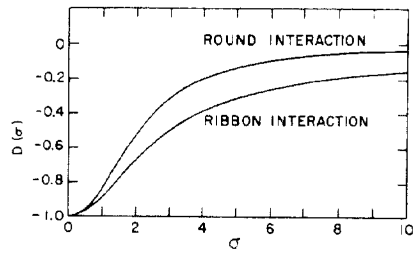

Fig. 1. Detuning functions, $D(\sigma)$, for ribbon and round interaction regions.

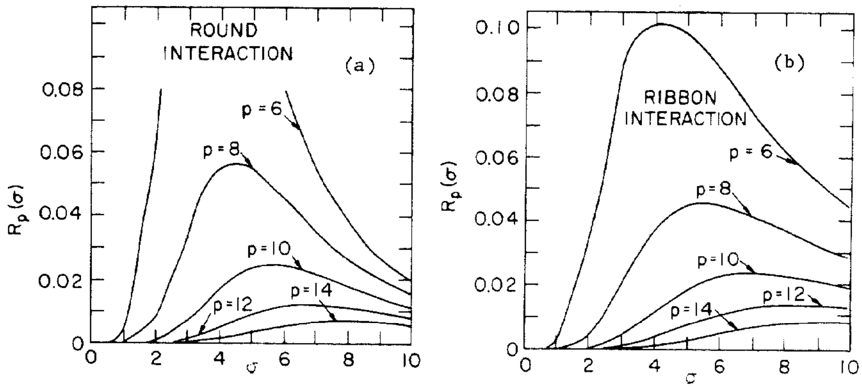

Fig. 2. Resonance functions, $R_p(\sigma)$, for even-ordered resonances. (a) Round interaction. Maximum, $R_6(3) = 0.15$. (b) Ribbon interaction.

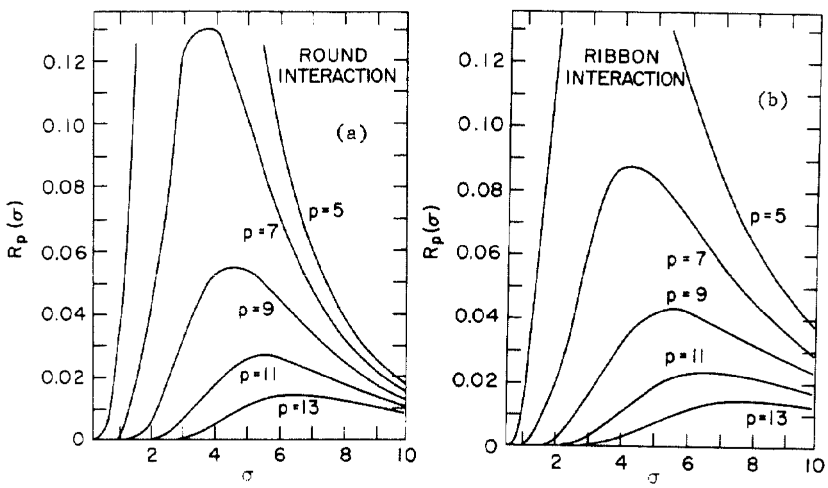

Fig. 3. Resonance functions, $R_p(\sigma)$, for odd-ordered resonances. (a) Round interaction. Maximum, $R_5(3) = 0.38$. (b) Ribbon interaction. Maximum, $R_5(3) = 0.22$.

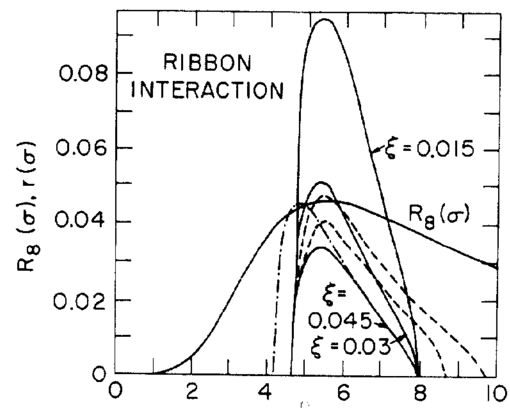

Fig. 4. Trapping for the 8th order resonance with a ribbon interaction.

— $\bar{\nu}/\xi=-0.67, k=3.0, \xi=0.015, 0.03, 0.045$
--- $\bar{\nu}/\xi=-0.67, k=3.5, 4.0, \xi=0.045$
-·- $\bar{\nu}/\xi=-0.74, k=3.5, \xi=0.045$.

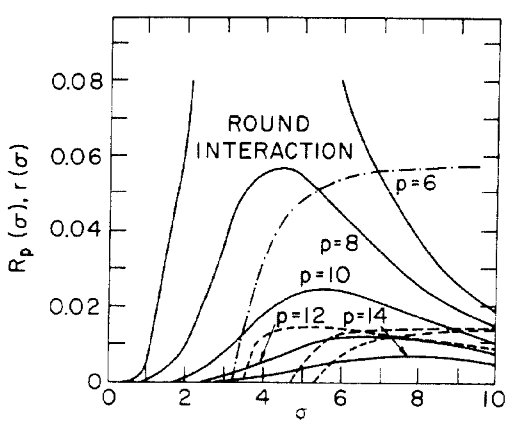

Fig. 5. Trapping for even-ordered resonances with a round interaction.

--- $\xi=0.03, k=3.0, \bar{\nu}_o=0, -0.002, -0.008$
-·- $\xi=0.015, k=3.0, \bar{\nu}_o=0$.

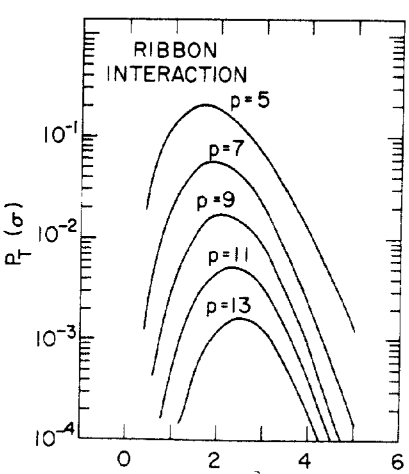

Fig. 6. Trapping probability function for odd-ordered resonances and a ribbon interaction. $\langle\sigma\rangle_{rms}=1$.

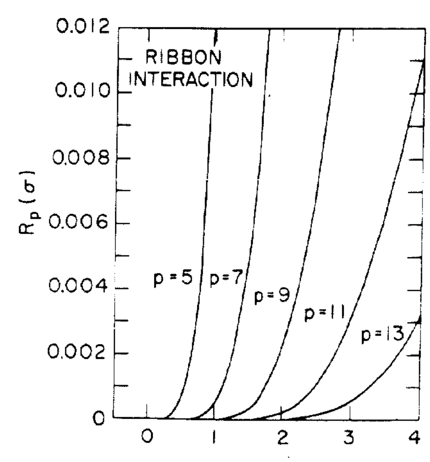

Fig. 7. Resonance functions, $R_p(\sigma)$, for odd-ordered resonances and a ribbon interaction. Small σ dependence.