BEAM LOADING OF RF CAVITIES

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Abstract

An analysis of the microwave properties of an RF cavity, including the effects of beam interaction, is given. The analysis includes the effect of input coupling mis-match and transient conditions. The principal result is that the energy gain must be represented by a cubic expression rather than the quadratic form applicable to traveling wave structures.

When the attenuation of the structure is negligible in establishing the energy density along its length, the repartition of the input power, P_o, is given by the differential equation

$$P_o = dW/dt + P_r + i \cdot V \qquad (1)$$

where W is the stored energy, P_r is the joulean loss in the structure walls and $i{\cdot}V$ is the power abstracted by the beam. By definition of the shunt impedance per unit length, r_o,

$$r_o/Q_o = E^2/2\omega w = E^2 L/2\omega W \qquad (2)$$

where E is the electric field intensity and w is the energy density per unit length, the "voltage" across the cavity (or tank) is given by

$$V = \left(2 r_o \omega W L/Q_o\right)^{1/2} \qquad (3)$$

Since the loaded Q is defined by the ratio of energy stored to the joulean losses per radian, $Q_L = \omega W/P_r$ Eq (1) becomes

$$dW/dt + \omega W/Q_L + i\left(2 r_o \omega W L/Q_L\right)^{1/2} - P_o = 0$$

or, since we are interested in the system energy gain,

$$(Q_o V/r_o \omega L)\, dV/dt + Q_o V^2/2 Q_L r_o L + iV - P_o = 0 \qquad (5)$$

In the steady state (dV/dt = 0) the resultant quadratic equation has the solution

$$V = \left(2 P_o r_o L\, Q_L/Q_o + \left(i r_o L\, Q_L/Q_o\right)^2\right)^{1/2} - \left(i r_o L\, Q_L/Q_o\right) \qquad (6)$$

Unfortunately from the point of view of simplicity, the coupling coefficient depends upon the beam loading so that the power input to the cavity is not the incident power, P_i. However, for ulterior reasons, suppose that critical coupling were achievable at all beam currents. Then, $P_o = P_i$ and Eq (6) becomes

$$V = \left(P_i r_o L + (i r_o L/2)^2\right)^{1/2} - (i r_o L/2) \qquad (7)$$

which may be put in the form, where $\eta = iV/P_i$ is the beam power conversion efficiency,

$$V^2/P_i r_o L = 1 - \eta \qquad (8)$$

shown (dashed) in Figure 1.

The reader may, perhaps see an apparent contradiction between the result, Eq (7), and the definition of shunt impedance: a factor of two seems to have been lost. The explanation is, of course, that the "loaded" or coupled shunt impedance is appropriate in Eq (7). That is, $r_L/Q_L = r_o/Q_o$, assuming coupling does not cause a repartition of energy in the cavity.

When a gap load, such as a beam, exists in a cavity the figures of merit are modified, such that without loading $Q_o/Q_L = 1 + \beta$ and with the loading $Q_o'/Q_L' = 1 + \beta'$ If the loading is purely resistive (R_1) then

$$R_1/Q_o' = (R_o + R_1)/Q_o \qquad (9)$$

Since the external Q of the system does not change, $Q_e = Q_e'$ and

$$Q_L' = Q_o R_1/\left(R_o + (1+\beta) R_1\right) \qquad (10)$$

Similarly,

$$\beta' = \beta R_1/(R_o + R_1) \qquad (11)$$

Eqs. (9), (10), and (11) give the relations between the loaded and open circuit cavity characteristics, such that if the properties of the cavity are known its properties with a gap load can be determined.

In the present case the beam loading $R_1 = V/i$ and the shunt impedance $R_o = r_o L$. Thus, if an open-circuit coupling coefficient has been chosen the coupling at any other beam loading is given by

$$\beta = \beta_{oc}\, V/(V + i r_o L) \qquad (12)$$

Observing that the power input to the cavity is

$$P_o = (1 - |\rho|^2) P_i = 4\beta P_i/(1+\beta)^2 \qquad (13)$$

where the reflection coefficient $|\rho| = (\sigma - 1)/(\sigma + 1)$ and the standing wave ratio σ is equal to the coupling coefficient at resonance (or its reciprocal), the system energy gain, Eq (6), can then be put in the form (by squaring)

$$V^2 + 2 i r_o L\, V/(1+\beta_{oc}) = 8 P_i r_o L \beta/(1+\beta_{oc})(1+\beta)^2 \qquad (14)$$

The energy gain of the system is now clearly more complex; inserting the value of the coupling coefficient, Eq (12), into the energy gain equation, (Eq (14), results in a cubic equation (in standard form)

$$V^3 + a_2 V^2 + a_1 V + a_o = 0 \qquad (15)$$

where the coefficients:

$$a_2 = 4 i r_o L/(1+\beta_{oc})$$

$$a_1 = \left[5 (i r_o L)^2 (1+\beta_{oc}) - 8 P_i r_o L \beta_{oc}\right]/(1+\beta_{oc})^3$$

$$a_o = \left[2 (i r_o L)^3 - 8 P_i r_o L \beta_{oc} (i r_o L)\right]/(1+\beta_{oc})^3$$

The solution of Eq (15) may be performed directly, taking advantage of the physical constraints to eliminate non-pertinent solutions. A second method is the use of iteration; that is, one can compute the energy gain using the open-circuit coupling coefficient is calculated, the cycle being repeated until the values converge.

Two points along the solution of Eq (15), in particular, serve to resolve ambiguities in the solution of that equation, in addition to the design point. Specifically, at no-load (i = o) the energy gain is given by

$$V_o^2 = 8 P_i r_o L \beta_{oc}/(1+\beta_{oc})^3 \qquad (16)$$

When the energy gain vanishes, the beam current is given by $a_o = 0$, or

$$i_m^2 = 4 P_i \beta_{oc}/r_o L \qquad (17)$$

Eq (15) can be put in the form of a universal diagram, since $i r_o L/V = \eta P_i r_o L/V^2$ where $\eta = iV/P_i$ is the beam power conversion efficiency. Then the energy gain equation,

$$\left(V + i r_o L/(1+\beta_{oc})\right)^2 \left(V + 2 i r_o L/(1+\beta_{oc})\right) \qquad (18)$$

$$= 8 P_i r_o L \beta_{oc} (V + i r_o L)/(1+\beta_{oc})^3$$

may, by algebraic manipulations, be cast in the form

$$\left(\frac{P_i r_o L}{V^2}\right)^3 + \left(\frac{P_i r_o L}{V^2}\right)^2\left(\frac{5(1+\beta_{oc})}{2\eta} - \frac{4\beta_{oc}}{\eta^2}\right) + \left(\frac{P_i r_o L}{V^2}\right)\cdot\left(2\left(\frac{1+\beta_{oc}}{\eta}\right)^2 + \frac{4\beta_{oc}}{\eta^3}\right) + \frac{1}{2}\left(\frac{1+\beta_{oc}}{\eta}\right)^3 = 0$$

the solution of which is shown in Figure 1. These results, while not particularly surprising, are somewhat unfamiliar and some comment is therefore appropriate.

Superficially, it would appear that a matched cavity ($\beta = 1$) would exhibit the greatest gap voltage for a given incident power from a matched source; that is, in fact, not true and a cavity matched to $\beta = 0.5$ will have a greater no-load gap voltage, for the reason that, while less incident power is transmitted through the coupling mechanism, the shunt resistance (or the loaded Q) is lowered to a lesser degree. The open-circuit gap voltage, taken from Eq (16), is shown in Figure 2.

On the other hand, with beam loading it is evident that the (quadratic) expression for the energy gain, Eq (7), is far from being accurate. This is due to the fact that beam loading affects the stored energy in the cavity and the coupling coefficient but not the joulean losses; thus, the open-circuit coupling coefficient enters into the power flux equation in two ways.

In the above discussion we have ignored the effect of beam loading on the resonant frequency of the cavity, since the frequency of the normal mode of oscillation ω_o will be perturbed (downward) due to the reduction in the cavity Q. However, the operator will adjust the driving frequency whtiout especially noticing the effect of loading and further discussion is omitted.

Optimum Designs

In principle the energy gain equation Eq (15) can be solved for the open circuit coupling coefficient to achieve a desired energy level and beam current with the structure shunt impedance and power input available. However optimal design implies a maximization process. Thus, maximizing the output beam power with respect to the beam current and insert this condition in the energy gain equation one obtains a solution illustrated in Figure 3. Similarly, maximizing energy gain with respect to the choice of coupling coefficient for a specified beam loading and reinserting this expression in the energy gain equation, a solution is obtained which is also plotted in Figure 3. Because of the lengthy and tedious algebraic operations the details have been omitted.

Note that the universal diagram, Figure 1, implies that nearly constant energy gain can be obtained over a wide range of beam loading, but at a cost in the energy level achievable. Further, with higher coupling coefficients the no-load energy gain can be less than with beam loading.

Transient Regime

The energy gain of the waveguide during the transient regime depends, of course, on the time of beam injection with respect to the commencement of the rf signal. With no beam loading the differential equation for the stored energy is given by Eq (4),

$$dW/dt + \omega W/Q_L - P_o = 0$$

which has the solution, with the boundary conditions $W = 0$, $t = 0$

$$W = (P_o Q_L/\omega)(1 - e^{-\omega t/Q_L})$$

or, by Eq (3), the "voltage" across the tank is

$$V^2 = 8 P_i r_o L \beta_{oc}\left(1 - e^{-\omega t(1+\beta_{oc})/Q_o}\right)/(1+\beta_{oc})^3 \qquad (19)$$

If the beam injection occurs at a time t_o after the commencement of the rf pulse the energy gain transient is given by the solution of Eq (5) using the solution of Eq (19) at time t_o as a boundary condition. The solution of Eq (5) is, however, somewhat more complicated than at first appearance, because the coupling coefficient is now also a variable.

Substituting Eqa (9), (10) and (11) as appropriate into Eq (5) we have

$$(2Q_o/\omega)\,dV/dt = -\frac{(a_3 V^3 + a_2 V^2 + a_1 V + a_o)}{(i r_o L + (1+\beta_{oc})V)^2} \qquad (20)$$

where

$a_3 = (1+\beta_{oc})^3/2$

$a_2 = 2(1+\beta_{oc})^2 i r_o L$

$a_1 = \frac{5}{2}(1+\beta_{oc})(i r_o L)^2 - 4 P_i r_o L \beta_{oc}$

$a_o = (i r_o L)^3 - 4 P_i r_o L \beta_{oc}(i r_o L)$

Integrating this expression, with the boundary conditions $V = V_o$, $t = t_o$

$$\frac{(i r_o L)^2 + (1+\beta_{oc}) i r_o L V_1 - \frac{8 P_i r_o L \beta_{oc}}{1+\beta_{oc}}}{3a_3 V_1^2 + 2a_2 V_1 + a_1} \ln\frac{V - V_1}{V_o - V_1} - \ln\frac{a_3 V^3 + a_2 V^2 + a_1 V + a_o}{a_3 V_o^3 + a_2 V_o^2 + a_1 V_o + a_o} = \frac{3\omega(t - t_o)}{2Q_o} \qquad (21)$$

where V_1 is the steady state solution ($t \to \infty$). As an alternate, in view of the implicit nature of the solution, it is also convenient to solve Eq (20) numerically.

Effect of Frequency Error

Unlike the travelling wave mode of operation, which is a band pass filter, resonant operation of a waveguide permits the structure to be excited only at discrete frequencies in the pass-band and the waveguide will generally exhibit n + 1 resonances, where n is the number of periodic lengths of the structure.

The amplitude of the cavity response to excitation is proportional to the real part of its impedance, that is, the shunt resistance. When the resonant (radian) frequency is ω_o the "voltage" across the cavity tank at steady state, without beam loading, is given by

$$V^2 = \frac{8 P_i r_o L \beta_{oc}}{(1+\beta_{oc})^3}\cdot\frac{1}{1+\left(2Q_L\frac{\omega-\omega_o}{\omega+\omega_o}\right)^2} \qquad (22)$$

and in the presence of loading by a considerably more complicated expression, since shunt resistance enters the energy gain equation, Eq (15), in the third degree. Obviously a simple way to determine the energy decrement due to mis-tuning is to solve Eq (15) for the energy gain using several values of shunt resistance corrected for the frequency error.

In addition to the reduced cavity voltage caused by off-resonance operation, there is another energy decrement owing to desynchronization between the beam and the cavity excitation. The phase discrepancy per cavity, $\Delta\phi = -Q(\omega - \omega_o)/\omega_o$ thus, if there are n cavities per unit length, the total energy gain, where V_o is the maximum realizable at synchronism, can be easily shown to be given by

$$\frac{V}{V_o} = \frac{\sin(Q(\omega-\omega_o)nL/\omega)}{Q(\omega-\omega_o)nL/\omega} \qquad (23)$$

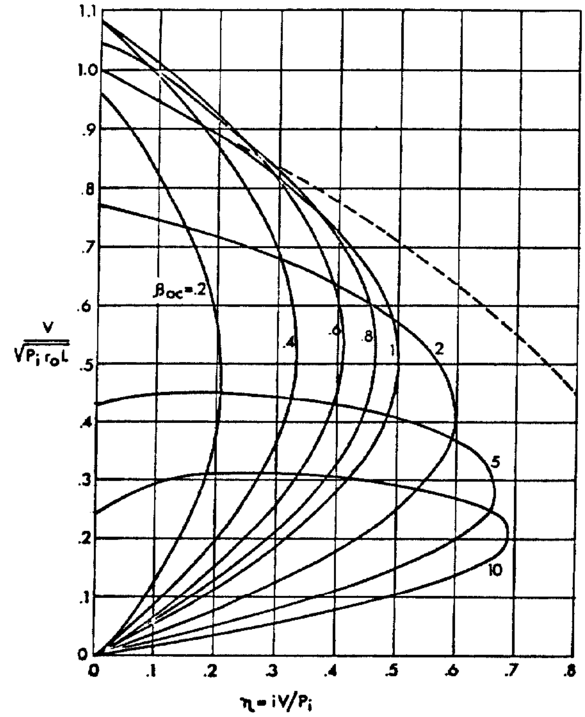

FIGURE 1. PLOT OF ENERGY GAIN EQUATION, EQ.(15)

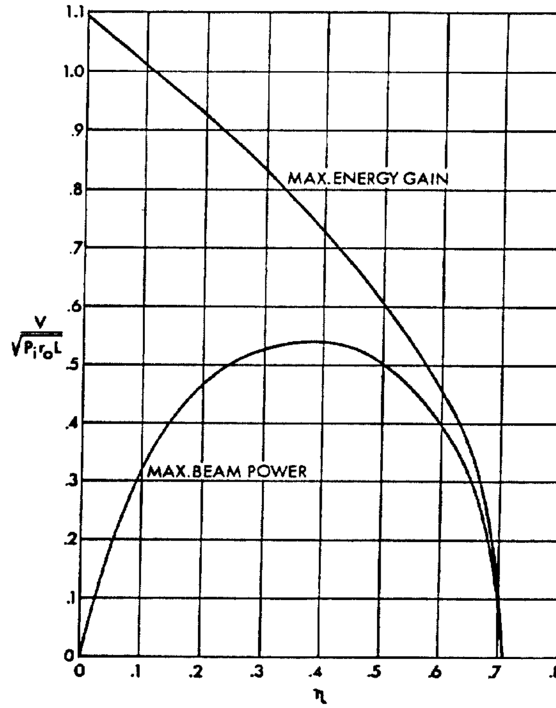

FIGURE 3. OPTIMUM DESIGNS FOR BEAM INTERACTION CAVITIES.

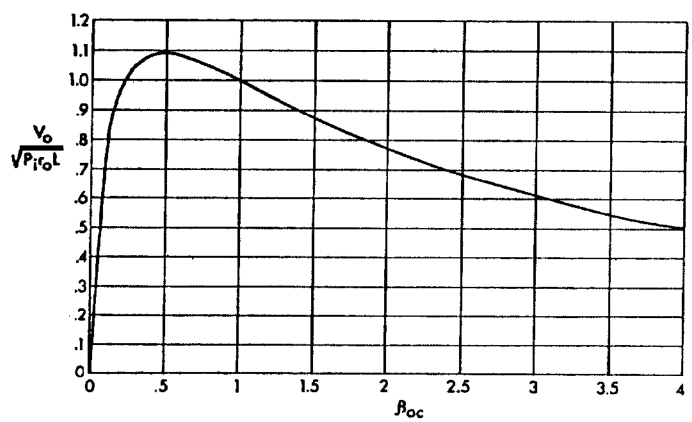

FIGURE 2. NO LOAD GAP VOLTAGE AS A FUNCTION OF INPUT COUPLING FACTOR.