EQUIVALENT CIRCUIT FOR A PERTURBED MULTI-CELL STRUCTURE

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I. Introduction

The purpose of this paper is to determine the appropriate equivalent circuit for a multi-cell structure and to compare its predictions for dispersion curves and sensitivity to perturbations with those obtained with less rigorously derived equivalent circuits.

We first derive an equivalent circuit and a dispersion relation for a periodic Alvarez type linear accelerator using the approach of Bevensee.[1] This approach starts from Maxwell's equations with the fields expanded in the normal modes appropriate to the structure. The equivalent circuit that these equations imply (called circuit 1) is different from the equivalent circuit (called circuit 2) used by many other authors.[2,3,4]

While many circuits lead to a single passband dispersion relation, it is not obvious that they each give an identical dispersion curve. If these circuits are modified to allow for perturbations, they may not in general predict the same frequency shift and the same changes in the gap fields for the same perturbations.

In order to compare circuit 1 with circuit 2, we compute the frequency shift and the changes in the gap fields using a multi-cell field formulation.[5] It will be shown that the two circuit models are not equivalent, and that the Bevensee model with a frequency dependent coupling parameter gives more accurate predictions for the perturbed quantities.

II. Equivalent Circuit

The geometry we are considering is the azimuthally symmetric structure shown in Fig. 1 where we have chosen the unit cell to be the one within the dashed lines. We define the first short-circuit cavity mode to be the solution of

$$\nabla \times \bar{E}_1 = P_1 \bar{H}_1 \ , \nabla \times \bar{H}_1 = P_1 E_1, \ P_1 = \omega_1 \sqrt{\mu\varepsilon} \qquad (II.1)$$

with the boundary condition $E_1 \times \hat{n}=0$ on metal surfaces and coupling surfaces (the dashed lines show the coupling surfaces). Similarly, we define the open-circuit cavity mode to be the solution of

$$\nabla \times \bar{e}_1 = p_1 \bar{h}_1 \ , \nabla \times \bar{h}_1 = p_1 \bar{e}_1, \ p_1 = \omega_1' \sqrt{\mu\varepsilon} \qquad (II.2)$$

with the boundary condition $\bar{e}_1 \times \hat{n}=0$ on metal surfaces and $e_1 \cdot \hat{n}=0$ on the coupling surfaces. The fields are normalized such that

$$\int_V \bar{E}_1^2 dV = \int_V \bar{H}_1^2 dV = \int_V \bar{e}_1^2 dV = \int_V \bar{h}_1^2 dV = \tau(\text{cavity volume}) \qquad (II.3)$$

Now we want to solve Maxwell's equations in a unit cell with $\omega\sqrt{\mu\varepsilon} \tilde{\simeq} P_1$ (the implications of this approximation on the passband parameter will be discussed later). If the other resonant cavity modes have resonant frequencies $P_2<P_3<..$, such that $P_2-\omega\sqrt{\mu\varepsilon} \gg P_1-\omega\sqrt{\mu\varepsilon}$, then only the first resonant cavity mode will be strongly excited. Thus we will write the fields as $\vec{E}=V_1\bar{E}_1$ and $\vec{H}=I_1\bar{H}_1$ leaving V_1 and I_1 to be determined. To determine V_1 and I_1 we write Maxwell's equations in the absence of beam current and power inputs and following Bevensee we get the two equations

$$\tau P_1 V_1 + \int \vec{E}_{tan} \times \bar{H}_1 \cdot \hat{n} dS = -j\omega\mu I_1 \tau, \qquad (II.4)$$

(where the integral is over all coupling surfaces)

$$P_1 I_1 = j\omega\varepsilon V_1 \qquad (II.5)$$

We now wish to write the tangential electric field in Equation (II.4) more explicitly. Since the short-circuit mode has no tangential electric field at the coupling surfaces, we will want to use the open circuit modes to write $\vec{E}_{tan}$. Following Bevensee, we write for the n^{th} cell on the surface M_+

$$\vec{E}_{tan} \tilde{=} \frac{1}{2}\bar{e}_1 V_1^n - \frac{1}{2}\bar{e}_1 V_1^{n+1} \qquad (II.6)$$

where $\vec{E}_{tan}$ reduces to the appropriate value for both the 0 mode ($V_1^n = V_1^{n+1}$, $E_{tan}=0$) and π mode ($V_1^n = -V_1^{n+1}$, $\vec{E}_{tan}=\bar{e}_1 V_1^n$). Note that the field pattern e_1 is repeated from cell to cell while the amplitudes are phase shifted from cell to cell. Since our expression for $\vec{E}_{tan}$ is correct at both ends of the passband, it should be a good approximation for the intermediate modes and frequencies also.

With the expression (II.6), we are in a position to write equation (II.4) in a more meaningful form. Remembering that $\hat{n}=\hat{i}_z$ on M_+ and $\hat{n}=-\hat{i}_z$ on M_- we can write equation (II.4) for the n^{th} cell as (dropping the subscripts)

$$P_1 V^n = -\frac{1}{\tau}\frac{V^n}{2}\int_{M_+}\bar{e}\times\bar{H}\cdot\hat{i}_z dS + \frac{1}{\tau}\frac{V^{n-1}}{2}\int_{M_+}\bar{e}\times\bar{H}\cdot\hat{i}_z dS + \frac{1}{\tau}\frac{V^n}{2} \times$$
$$\times \int_{M_-}\bar{e}\times\bar{H}\cdot\hat{i}_z dS - \frac{1}{\tau}\frac{V^{n+1}}{2}\int_{M_-}\bar{e}\times\bar{H}\cdot\hat{i}_z dS - j\omega\mu I^n \qquad (II.7)$$

We define the passband parameter

$$M = \frac{1}{\tau}\int_{M_-}\bar{e}\times\bar{H}\cdot\hat{i}_z dS \qquad (II.8)$$

and note that $\bar{e}|_{M_+} = -\bar{e}|_{M_-}$. Then we can write

$$P_1 V^n = V^n M - \frac{1}{2}(V^{n-1} + V^{n+1})M - j\omega\mu I^n. \qquad (II.9)$$

Here we will introduce the definitions: $V_n=V^n/P_1$, $I_n=I^n/P_1$, $L_n=\mu/P_1$, $C_n=\varepsilon/P_1$. Note that V_n and I_n refer to voltages and currents while V^n and I^n refer to field amplitudes. We can now write

$$I_n = j\omega C_n V_n \qquad (II.10)$$

$$V_n = \frac{I_n}{j\omega C_n}\frac{M}{P_1} - \frac{1}{2j\omega C_n}\frac{M}{P_1}(I_{n-1} + I_{n+1}) - j\omega L_n I_n. \qquad (II.11)$$

A comment is necessary here on the passband parameter M as defined in Eq.(II.8). In a more general formulation than that used by Bevensee, M can be expected to have frequency dependence. This is evident since $\bar{e}$ resonates at the frequency characteristic of the first open-circuit mode, while $\bar{H}$ resonates at the frequency characteristic of the first closed circuit mode. We will subsequently generalize M accordingly. However, for a narrow passband ($\omega\sqrt{\mu\varepsilon} \tilde{\simeq} P_1$), these frequencies are approximately equal. Now equations (II.10) and (II.11) are represented by the circuit in Figure 2.a where we have

$$Z_1 = j\omega L_n \ , \ Z_2 = 1/j\omega C_n \qquad (II.12)$$

If we consider k to be independent of ω as Bevensee does, then we have

$$Z_c = -k/2j\omega C_n \qquad (II.13)$$

and the circuit is that shown in Figure 2.b. However, if we allow ω dependence in k, then a convenient form for Z_c is one which corresponds to a coupling element which can be made to resonate, i.e.

$$Z_c = j\omega k_L L_n/2 - k_C/2j\omega C_n \qquad (II.14)$$

and our circuit is that shown in Figure 2.c.

Here we will consider the circuit in Figure 2.b. The circuit equations are (II.10) and (II.11).

Combining these we get

$$(1-\omega_a^2/\omega^2)I_n=k\omega_a^2(I_{n+1}+I_{n-1}-2I_n)/2\omega^2;\omega_a^2=1/L_nC_n;$$
$$\omega_a/c = P_1;\ k = M/P_1 \qquad (II.15)$$

If we now consider an N-cell structure terminated by half-cells with metallic boundaries, it is simplest to use the equivalent boundary conditions $I_{-1} = I_1$, $I_{N-1} = I_{N+1}$, leading to the solution

$$I_n^q = \cos n\phi_q,\ \phi_q = \pi q/N,\ q = 0,1,\ldots,N \qquad (II.16)$$

$$(I^q,I^q)=\sum_n W(n)I_n^qI_n^q = \frac{N}{2W(q)};\ \frac{W(q)}{W(n)} = \begin{cases} 1/2 & q=0,N \\ & n=0,N \\ 1 & q=1,2,3,\ldots,N-1 \\ & n=1,2,3,\ldots,N-1 \end{cases} \qquad (II.17)$$

and to the dispersion relation

$$1-\omega_a^2/\omega_q^2 + (k\omega_a^2/\omega_q^2)(1 - \cos\phi_q) = 0 \qquad (II.18)$$

In Eq.(II.18) ω_q is the frequency associated with ϕ_q. For the structure we are considering, Eq.(II.15) is correct for n=1,2,3,...,N-1. For n=0,N the equations are

$$(1-\omega_a^2/\omega_q^2)I_o^q = (k\omega_a^2/\omega_q^2)(I_1^q - I_o^q) \qquad (II.19)$$

$$(1-\omega_a^2/\omega_q^2)I_N^q = (k\omega_a^2/\omega_q^2)(I_{N-1}^q - I_N^q) \qquad (II.20)$$

The above equations with equation (II.15) can be summarized in the matrix equation

$$M_qI^q = 0\ . \qquad (II.21)$$

III. Determination of Perturbed Parameters

We now wish to determine how a perturbation in our structure affects the frequency and the gap fields. The following discussion outlines our method of obtaining the unperturbed circuit parameters $\omega_a=(LC)^{-\frac{1}{2}}$ and k and the perturbed circuit parameters $\bar{\omega}_a=(\bar{L}\bar{C})^{-\frac{1}{2}}$ and $\bar{k}$.

We have two formulations of the field calculation program.[5,6] The formulation in reference 6 is an accurate and simple way to obtain frequencies for periodic geometries. These frequencies are then used as a starting point to find the frequencies and gap fields in the formulation of reference 5. In our calculations, we used a five cell periodic structure terminated in half-cells to determine our circuit parameters and then we introduced a perturbation in the first half-cell by increasing the gap size by 1%, .5%, and .1%.

We determined the circuit parameters ω_a and k by using an initial geometry and calculating the ω_o and ω_π frequencies and then putting these in the dispersion curve to calculate ω_a and k. We then went to another periodic geometry which differed from our initial geometry only in that all the gaps had been increased by 1%, .5%, or .1%. We calculated new values of ω_o and ω_π, and from these we obtained values for $\bar{\omega}_a$ and $\bar{k}$ corresponding to the different geometries. At this point we introduced the gap perturbation into the first half cell only and calculated a new frequency for the structure as well as the new gap fields.

IV. Perturbations on the Bevensee Model

In order to investigate the effects of a perturbation in the circuit equations, we introduce the new circuit parameters $\bar{L}$, $\bar{C}$, and $\bar{k}$ into the first half-cell of our circuit and keep the remaining cells unchanged, i.e., with circuit parameters L, C, and k. In this model we take the coupling element between perturbed and unperturbed cells to be the average of the coupling element for each cell. We then calculate to first order what will be the frequency shift and the changes in the gap fields. Thus, the circuit we will investigate is shown in Figure 4.a. Our perturbed equations to first order are

$$[1-\frac{\omega_a^2}{\omega^2}(1+\varepsilon_0-\Delta)]I_0'+\frac{k\omega_a^2}{\omega^2}(1+\frac{\Delta k}{2k}+\frac{\varepsilon_0}{2}-\Delta-\frac{\bar{\gamma}}{2})(I_0'-I_1')=0 \qquad (IV.1)$$

Similarly,we get

$$[1-\frac{\omega_a^2}{\omega^2}(1-\Delta)]I_1'+\frac{k\omega_a^2}{2\omega^2}(1+\frac{\Delta k}{2k}+\frac{\varepsilon_0}{2}-\Delta+\frac{\bar{\gamma}}{2})(I_1'-I_0') + \frac{k}{2}\frac{\omega_a^2}{\omega^2}(1-\Delta)(I_1'-I_2') = 0 \qquad (IV.2)$$

and

$$[1-\frac{\omega_a^2}{\omega^2}(1-\Delta)]I_n'-\frac{k}{2}\frac{\omega_a^2}{\omega^2}(1-\Delta)(I_{n+1}'+I_{n-1}'-2I_n')=0 \quad n\geq 2,\ n\neq N \qquad (IV.3)$$

$$[1-\frac{\omega_a^2}{\omega^2}(1-\Delta)]I_N'-k\frac{\omega_a^2}{\omega^2}(1-\Delta)(I_{N-1}'-I_N') = 0 \qquad (IV.4)$$

where

$$\bar{\omega}_a^2=(\bar{L}\bar{C})^{-1};\omega_a^2=(LC)^{-1};\gamma=\bar{L}/L=1+\delta L/L=1+\bar{\gamma}\ . \qquad (IV.5)$$

Note that the primed quantities refer to the values in the presence of the perturbation, while the barred quantities refer to the values due to an equal perturbation in every cell. Also, we have used

$$\bar{\omega}_a^2 = \omega_a^2(1+\varepsilon_0),\quad \varepsilon_0=(\delta\omega_a^2/\omega_a^2)_0 \qquad (IV.6)$$

and

$$\omega'^2=\omega_q^2(1+\Delta),\ \Delta = \delta\omega_q^2/\omega_q^2 \qquad (IV.7)$$

Now we consider the perturbation of the matrix equation $M_qI^q=0$ which is, to first order,

$$\delta M_q I^q + M_q\ I^q = 0 \qquad (IV.8)$$

Expanding δI^q as

$$\delta I^q = \sum_{r\neq q}\alpha_{qr}\ I^r \qquad (IV.9)$$

(which implies that the normalization of I' is the same as that of I) we get for the frequency shift and expansion coefficients

$$\delta\omega_q^2/\omega_q^2=-(\omega_a^2/\omega_q^2)(I^q,\varepsilon I^q)/(I^q,I^q);\quad \varepsilon = (\omega_q^2/\omega_a^2)(\delta M_q-\delta\omega_q^2/\omega_q^2) \qquad (IV.10)$$

and

$$\alpha_{qr} = -(I^r,\varepsilon I^q)/[(I^r,I^r)(1-\omega_r^2/\omega_q^2)] \qquad (IV.11)$$

Evaluating equations (IV.10) and (IV.11) explicitly for the zero phase shift mode for a perturbation in the zeroth cell we get (since $\omega_a^2 = \omega_0^2$)

$$\delta\omega_0^2/\omega_0^2 = \varepsilon_0/2N \qquad (IV.12)$$

and

$$\alpha_{00}=0,\alpha_{01}=\varepsilon_0/[N(1-\omega_1^2/\omega_0^2)],\alpha_{02}=\varepsilon_0/[N(1-\omega_2^2/\omega_0^2)],\text{etc.} \qquad (IV.13)$$

Note here that $\delta L/L$ does not appear in equation (IV.12) or equation (IV.13). This is fortunate since it means we don't have to treat L or C for a cavity separately, but we have only to deal with the product $LC(=1/\omega_a^2)$ which is easily measured (or computed). It is interesting to note that the only other mode where this cancellation occurs is the π-mode.

V. Previously Used Equivalent Circuit

Now we will consider the circuit shown in Figure 3 which is frequently used to represent a periodic

structure. The equations implied by this circuit are

$$(1-\omega_a''^2/\omega_q^2)I_o = -k''I_1 \; ; \; \omega_a''^2 = (L'C')^{-1} \qquad (V.1)$$

$$(1-\omega_a''^2/\omega_q^2)I_n = -(k''/2)(I_{n+1}+I_{n-1}) \; ; n=1,2,\ldots N-1 \qquad (V.2)$$

$$(1-\omega_a''^2/\omega_q^2)I_N = -k''I_N \; . \qquad (V.3)$$

The solution for I_N is the same as for the Bevensee equations, but the dispersion relation is now

$$1 - \omega_a''^2/\omega_q^2 + k''\cos q = 0 \; . \qquad (V.4)$$

In this model k" is taken to be independent of ω. As with the Bevensee equations, we can write equations (IV.1), (IV.2), and (IV.3) in the matrix form

$$M_q I^q = 0 \qquad (V.5)$$

VI. Perturbations in the Previously Used Model

Most authors have considered k to be constant when they have studied perturbations in equivalent circuits. It became evident early in our investigation that a perturbation corresponded to a change in k as well as in ω_a. We will incorporate this into our analysis of perturbations in the previously used model.

The perturbed circuit we are considering is shown in Figure 4.b. The perturbed cell has quantities $\bar{\omega}_a''$ and $\bar{k}''$ associated with it, while the unperturbed cells have quantities ω_a'' and k". We will consider the coupling parameter between a perturbed cell and an unperturbed cell to be $(\bar{k}''+k'')/2$, i.e., the average of the coupling parameters in the two cells. This is reasonable since we would get the full change in the coupling parameter if we perturbed both cells equally. So, keeping this in mind, the equations which the circuit in Figure 4.b. implies are

$$(1-\bar{\omega}_a''^2/\omega'^2)I_0' + [(\bar{k}''+k'')(1-\delta L''/2L'')/2]I_1' = 0 \qquad (VI.1)$$

$$(1-\omega_a''^2/\omega'^2)I_1' + [(\bar{k}''+k'')(1+\delta L''/2L'')/2]I_0' + (k''/2)I_2' = 0 \qquad (VI.2)$$

and

$$(1-\omega_a''^2/\omega'^2)I_n' + (k''/2)(I_{n+1}' + I_{n-1}') = 0 \; n\geq 2, n\neq N \qquad (VI.3)$$

$$(1-\omega_a''^2/\omega'^2)I_N' + k''I_N' = 0. \qquad (VI.4)$$

Using Equation (IV.8), we get

$$\delta\omega_q^2/\omega_q^2 = -(\omega_q^2/\omega_a''^2)(I^q,\varepsilon''I^q)/(I^q,I^q) \qquad (VI.5)$$

and

$$\alpha_{qr} = -(\omega_s^2/\omega_a''^2)(I^r,\varepsilon''I^q)/[(I^r,I^r)(1-\omega_s^2/\omega_q^2)] \qquad (VI.6)$$

Evaluating equations (VI.5) and (VI.6) explicitly for the zero mode, we get

$$\delta\omega_0^2/\omega_0^2 = [\varepsilon_0'' - (\omega_0^2/\omega_a''^2)\Delta k'']/2N \qquad (VI.7)$$

and

$$\alpha_{00}''=0, \alpha_{01}'' = \left[\frac{\varepsilon_0''}{2} - \frac{\omega_0^2}{\omega_a''^2}\frac{\Delta k''}{2}\left(\frac{1+\cos(\pi/N)}{N}\right)\right] \Big/ \left[\frac{N}{2}(\omega_0^2/\omega_1^2-1)\right], \qquad (VI.8)$$

$$\alpha_{02}'' = \left[\frac{\varepsilon_0''}{2} - \frac{\omega_0^2}{\omega_a''^2}\frac{k''}{2}\left(\frac{1+\cos\frac{2\pi}{N}}{2}\right)\right] \Big/ \left[\frac{N}{2}(\omega_0^2/\omega_2^2-1)\right], \text{ etc.}$$

Note that, just as for the Bevensee model, the δL"/L" has cancelled out.

At this point, we can compare equation (VI.7) with equation (IV.12). Since we are fitting the dispersion curves for each model to the same ω_0 and ω_π, and since we have assumed that the coupling parameters k and k" are independent of ω in the two models, we can write

$$\omega_a^2 = \omega_a''^2/(1+k''), \; k = -k''/(1+k''). \qquad (VI.9)$$

Now using equation (IV.6) we can write

$$\varepsilon_0 = \varepsilon_0'' - (\omega_0^2/\omega_a''^2)\Delta k'' \; . \qquad (VI.10)$$

Thus, comparing equation (VI.7) and equation (IV.12), we see that they predict exactly the same frequency shift.

We are also able to compare the predictions that the two models give for the changes in the gap fields. When we compare equations (VI.8) with equations (IV.13), it is apparent that they agree approximately for a narrow passband and large N. We shall see, however, that while each model has certain regions of approximate validity, the model which gives best agreement in all cases is one based on Bevensee, allowing a frequency dependence in the coupling parameter k.

VII. Bevensee Model with Frequency Dependent k.

It is mentioned in section II that the passband parameter can be expected to have some frequency dependence. We have thus considered k to have the form

$$k = (\omega_q^2/\omega_a^2)k_L + k_C \qquad (VII.1)$$

where k_L is the coupling parameter associated with inductive coupling and $-k_C$ is the coupling parameter associated with capacitive coupling. Assuming this form for k will give us different results in the Bevensee formulation for the changes in the gap fields. The circuit for this case is shown in Figure 2.c. It will not change the prediction for the frequency shift. We then obtain

$$\delta\omega_q^2/\omega_q^2 = -(\omega_a^2/\omega_q^2)(I^q,\varepsilon I^q)/(I^q,I^q) \qquad (IV.10)$$

The expansion coefficients change because instead of

$$M_q I^r = (1 - \omega_r^2/\omega_q^r)I^r \qquad (VII.2)$$

$$M_q I^r = (1 - \omega_r^2/\omega_q^2)[1+k_L(1-\cos\phi_r)]I^r \qquad (VII.3)$$

from which we get the expansion coefficients

$$\alpha_{qr} = -(I^r,\varepsilon I^q)/\{(I^r,I^r)(1-\omega_r^2/\omega_q^2)[1+k_L(1-\cos\phi_r)]\} \qquad (VII.4)$$

We again compare the predictions for the frequency shift and changes in the gap fields. Since we choose k_L and k_C to be constants for a particular geometry, we can relate the quantities ω_a, ω_a'', k_L, k_C, and k" as in section VI, i.e.,

$$\omega_a^2 = \omega_a''^2/(1+k''), \; -k'' = (k_L+k_C)/[1+k_L-k_C] \qquad (VII.5)$$

From equation (IV.6), we find that equation (VI.10) is still valid and hence the frequency shift predicted by the two models is identical. It is evident that equations (IV.11), (VI.8) and (VII.4) give different results. We are now ready to determine which model agrees best with the results derived from the field model.[5]

VIII. Numerical Results and Discussion

We consider several geometries (of which Figure 1 is a typical example) which have differently spaced passbands and stopbands. This enables us not only to compare equivalent circuit models, but also to determine the influence of the second passband. The geometries we chose differ only in one parameter (the cell length), which is adjusted to control the location of the next passband. The numerical results are shown in Tables I, II, and III. Each geometry is characterized by a parameter α which is defined as

$$\alpha = \text{passband width/stopband width.} \qquad (VIII.1)$$

So the geometry with α=1/4 is one which has the second passband widely separated from the first passband, while the geometry with α=∞ has the two passbands confluent at the π-mode. The heading F.C. indicates the numbers derived from the field calculation.[5] The column headed BEVLSQ gives the numbers predicted by the Bevensee model with a frequency dependent coupling parameter. We fit the dispersion curve by the method of least squares to determine k_L and k_C. The parameter ω_a^2 is given by the dispersion curve to be ω_0^2. The third column gives the results of the Bevensee model with k independent of ω. It is labeled C because it represents a model with capacitive coupling. The last column gives the results of the previously used circuit model. It is labeled L because it represents a model with inductive coupling. Hopefully, the tables are self-explanatory.

It is interesting to note that the geometries we have chosen would seem to be inductively coupled since the coupling region is a region in which one expects mostly magnetic fields. However, we see from the values of k_L and k_C as a function of α, that coupling goes from essentially capacitive for $\alpha=1/4$ to essentially inductive for $\alpha=\infty$ with mixed coupling for $\alpha=1/2$. The reason for this is that, unexpectedly, there are modes for which the $\bar{E}$ field is large in the coupling region. (We see this when one of the resonant modes is anti-symmetric in the gap.)

When we examine the frequency shifts, we see that indeed the three equivalent circuit models give the same result. Also, the results of the equivalent circuit models agree very well with those of the field model. In fact, the results differ essentially by an amount proportion to $(\delta g/g)^2$.

An examination of the results for the gap field changes shows that the Bevensee model with an ω dependent coupling parameter gives the best agreement. When we look at the predictions for the dispersion curve, we see that this same Bevensee model also consistently fits the dispersion curve well. Correlating this with the results for the gap fields, we conclude that only when the dispersion curve is well fit do we get accurate predictions for the changes in the gap fields.

The predictions for the gap field changes in the zeroth cell are not as good as those for changes in the other cells. To understand and correct this we consider what the parameter I_n should represent. If we want to make the statement that the stored energy be

$$\text{Stored energy} = (1/2)W(n)L_n I_n^2 \qquad \text{(VIII.2)}$$

then we should interpret I_n as $I_n=E_n\sqrt{\lambda_n}$ where E_n is the amplitude of the electric field in the n^{th} gap and λ_n is the wavelength associated with the resonant frequency, ω_a, characterizing the n^{th} cell. This leads to

$$\delta I_n/I_n = \delta E_n/E_n - (1/2)(\delta f_n/f_n) \quad . \qquad \text{(VIII.3)}$$

For our case, $\delta f_n=0$ except for n=0, i.e., in the first cell. When we make this correction, we get the results in Table IV for the gap fields in the zeroth cell. We see that they now agree quite well.

It is evident that the model becomes less accurate as $\alpha\to\infty$. However, structures are usually designed to have widely separated passbands if they are operating in the zero mode. Operation at a $\pi/2$ mode (compensated structures) corresponds essentially to a confluence of two π-modes. We will investigate perturbations of the π-mode in the next section.

IX. Equivalent Circuit for a π-Mode

The circuit we have described in section VII does not accurately describe the behavior of the fields associated with a π-mode in the presence of a perturbation. That circuit was derived on the basis of a field of even symmetry resonating at the first closed circuit frequency. While the π-mode we are describing is a field of even symmetry, it resonates at the frequency of the first open circuit mode. Hence we derive a circuit based on these characteristics. The resulting equations are

$$v_n = -j\omega L i_n \qquad \text{(IX.1)}$$

and

$$i_n=j\omega C_n v_n+(k/2j\omega L_n)(v_{n+1}+v_{n-1})+(k/j\omega L_n)\, v_n \qquad \text{(IX.2)}$$

One circuit which is implied by these equations is shown in Figure 5.a. Combining equations (IX.1) and (IX.2) we get

$$(1-\omega_a^2/\omega^2-k\omega_a^2/\omega^2)v_n=(k\omega_a^2/2\omega^2)(v_{n+1}+ v_{n-1}) \qquad \text{(IX.3)}$$

with the same conditions as below (I.18). The solution for v_n is

$$v_n^q = \cos n\phi_q \ , \phi_q=\pi q/N, \ q=0,1,\ldots,N \qquad \text{(IX.4)}$$

and

$$(v^q,v^q)=N/2W(q) \qquad \text{(IX.5)}$$

with the dispersion curve

$$1-\omega_a^2/\omega_q^2-(k\omega_a^2/\omega_q^2)(1+\cos\phi_q) = 0 \qquad \text{(IX.6)}$$

Now we wish to introduce a perturbation in the first half cell of our equivalent circuit (see figure 5.b). The resulting prediction for the change in frequency is

$$\delta\omega_\pi^2/\omega_\pi^2=\varepsilon_\pi/2N; \ \varepsilon_\pi=\delta\omega_a^2/\omega_a^2; \ \omega_a^2 = \omega_\pi^2 \qquad \text{(IX.7)}$$

So that no confusion arises, the term on the left hand side of equation (IX.7) is the relative change in the frequency of the π-mode for the whole structure. The term ε_π on the right hand side of the equation refers to the relative change in the frequency of the π-mode in the perturbed cell.

The prediction for the expansion coefficients in the perturbed fields is

$$\alpha_{qr}=(\varepsilon_\pi/2)/\{(I^r,I^r)(1-\omega_q^2/\omega_r^2)[1-k_L(1+\cos\phi)]\} \qquad \text{(IX.8)}$$

The numerical results predicted by this model are compared to those of the field theoretical model in Table V. The agreement is excellent. Also, we see that the sensitivity to error is indeed much less than for the zero mode.

X. Summary

We have presented an equivalent circuit model based on Maxwell's equations, a phenomenological model, and a model with a frequency dependent coupling parameter. These models have been used to investigate the effects of perturbations on the zero and π-mode for different geometries characterized by the ratio of passband width to stopband width. The results have been compared to the predictions of a multi-cell field formulation. We have shown that when the dispersion curve for a structure is well matched, the predictions for the frequency shifts and changes in the gap fields are accurate. In addition we have shown that it is necessary to have a frequency dependent coupling parameter to obtain matching of the dispersion curve for different geometries. We have also shown that is necessary to describe the 0 and π-modes by different circuits.

It is a simple process to use our method to determine the effects of perturbations in a structure. One must obtain the dispersion curve for the structure under consideration (either theoretically via a field calculation or experimentally). Then introduce the same perturbation into every cell. This gives a new periodic structure for which one must obtain only the zero and π-mode frequencies. The circuit parameters are then obtained for the unperturbed and perturbed structures by a least squares fit of the dispersion curve. One then uses equations (IV.6), (IV.10), and (VII.4) to obtain the frequency shifts and changes in the gap fields. Note that this procedure is only for a single perturbation. If one wishes to study the effects of different perturbations in a structure, then it is necessary to obtain the 0- and π- modes frequencies and circuit parameters corresponding to each different perturbation. The same procedure is followed for investigations of the π-mode. However, the appropriate equations are now (IX.7) and (IX.8).

References

*Work supported in part by the National Science Foundation.

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TABLE I

Geometric Parameters

N=5;α=1/4;a=10.0cm;b=10.16002cm;g=1.2cm;L=7.620015cm

	F.C.	BEVLSQ	C	L
φx5/π	f	f	f	f
0	1019.945	1019.945	1019.945	1019.945
1	1041.025	1041.035	1042.346	1035.764
2	1095.559	1095.524	1098.829	1080.937
3	1161.812	1161.710	1164.867	1145.900
4	1214.570	1214.549	1215.671	1207.995
5	1234.525	1234.605	1234.525	1234.525

Coupling Parameters

k_L= -.0327; k_C=-.1845; k = -.2325; k = .1886

Perturbation in Zeroth Cell

Frequency Shifts - Zero Mode

δg/g	f	f	f	f
0.000	1019.94584	1019.94584	1019.94584	1019.94584
-0.001	1019.93393	1019.93395	1019.93395	1019.93395
-0.005	1019.88590	1019.88627	1019.88627	1019.88623
-0.010	1019.82492	1019.82622	1019.82622	1019.82606

Gap Field Shifts

δg/g = -0.010

Cell No.	δI/|I|	δI/|I|	δI/|I|	δI/|I|
0	.016984	.017801	.016644	.020504
1	.008111	.008091	.007565	.011534
2	.000534	.000539	.000504	.001175
3	-.004829	-.004855	-.004539	-.006224
4	-.008045	-.008091	-.007565	-.010664
5	-.009117	-.009170	-.008574	-.012144

δg/g = -0.005

0	.008451	.008865	.008288	.010203
1	.004050	.004029	.003767	.005740
2	.000284	.000269	.000251	.000585
3	-.002384	-.002418	-.002260	-.003097
4	-.003985	-.004029	-.003767	-.005307
5	-.004517	-.004567	-.004270	-.006043

δg/g = -0.001

0	.001683	.001770	.001655	.002036
1	.000808	.000804	.000752	.001145
2	.000060	.000054	.000050	.000117
3	-.000471	-.000483	-.000451	-.000618
4	-.000791	-.000804	-.000752	-.001059
5	-.000897	-.000912	-.000852	-.001206

TABLE II

Geometric Parameters

N=5;α=1/2;a=10.0cm;b=10.16002cm;g=1.2cm;L=10.0cm

Dispersion Curve

	F.C.	BEVLSQ	C	L
φx5/π	f	f	f	f
0	961.797	961.797	961.797	961.797
1	980.785	980.665	986.029	978.200
2	1032.339	1032.559	1046.816	1025.464
3	1101.397	1102.491	1117.392	1094.600
4	1164.479	1164.714	1117.382	1162.080
5	1191.358	1190.043	1191.358	1191.358

Coupling Parameters

k_L= -.1181; k_C= -.0847; k = -.2672; k = .2108;

Perturbation in Zeroth Cell

Frequency Shifts - Zero Mode

δg/g	f	f	f	f
0.000	961.79735	961.79735	961.79735	961.79735
-0.001	961.78213	961.78215	961.78215	961.78215
-0.005	961.72051	961.72117	961.72117	961.72111
-0.010	961.64182	961.64450	961.64450	961.64427

Gap Field Shifts

δg/g = -0.010

Cell No.	δI/|I|	δI/|I|	δI/|I|	δI/|I|
0	.024140	.025862	.019628	.026316
1	.011408	.011755	.008922	.014136
2	.000667	.000784	.000595	.001341
3	-.006887	-.007053	-.005353	-.007799
4	-.011354	-.011755	-.008922	-.013282
5	-.012833	-.013323	-.010112	-.015110

δg/g = -0.001

0	.002385	.002572	.001952	.002613
1	.001136	.001169	.000887	.001404
2	.000078	.000078	.000059	.000133
3	-.000668	-.000701	-.000532	-.000817
4	-.001112	-.001169	-.000887	-.001361
5	-.001261	-.001325	-.001005	-.001543

TABLE III

Geometric Parameters

N=5;α=∞;a=10.0cm;b=10.160020cm;g=1.2cm;L=13.856892cm

Dispersion Curve

	F.C.	BEVLSQ	C	L
φx5/π	f	f	f	f
0	843.858	843.858	843.858	843.858
1	859.167	858.130	876.219	862.600
2	903.172	902.713	955.766	918.278
3	969.657	979.034	1045.764	1004.743
4	1050.708	1070.379	1113.261	1095.958
5	1137.987	1116.859	1137.987	1137.987

Coupling Parameters

k_L = -.2748; k_C = .1056; k = -.4093; k = .2904

Perturbation in Zeroth Cell

Frequency Shifts-Zero Mode

δg/g	f	f	f	f
0.000	843.85753	843.85753	843.85753	943.85753
-0.001	843.84089	843.84130	843.84130	843.84130
-0.005	843.77534	843.77623	843.77623	843.77618
-0.010	843.69112	843.69454	843.69454	843.69433

Gap Field Shifts

δg/g = -0.010

Cell No.	δI/|I|	δI/|I|	δI/|I|	δI/|I|
0	.031248	.037658	.015571	.025719
1	.014539	.017117	.007078	.013181
2	.000681	.001141	.000472	.001152
3	-.008929	-.010270	-.004247	-.007440
4	-.014656	-.017117	-.007078	-.012596
5	-.016551	-.019400	-.008021	-.014314

δg/g = -0.001

0	.003124	.003751	.001551	.002559
1	.001454	.001705	.000705	.001311
2	.000068	.000114	.000047	.000115
3	-.000893	-.001023	-.000423	-.000740
4	-.001466	-.001705	-.000705	-.001253
5	-.001655	-.001932	-.000799	-.001424

TABLE IV

Zero Cell Corrected for $I_n=E_n\lambda_n^{1/2}$

δg/g	α	F.C. $E_n\lambda_n^{1/2}$	E_n
-0.010	1/4	.016984	.017214	.017801
-0.001	1/4	.001683	.001712	.001770
-0.010	1/2	.024140	.025067	.025862
-0.001	1/2	.002385	.002493	.002572
-0.010	∞	.031248	.036691	.037658
-0.001	∞	.003124	.003654	.003751

TABLE V

Geometric Parameters

N=5;a=10cm;b=10.16002cm;g=1.2cm

α = 1/4; L = 7.620015cm α =1 ; L = 11.375 cm

	F.C.	BEVLSQ	F.C.	BEVLSQ
φx5/π	f	f	f	f
0	1019.945	1019.820	922.353	926.875
1	1041.025	1041.009	940.276	941.798
2	1095.559	1095.660	990.180	986.707
3	1161.812	1161.856	1061.119	1057.999
4	1214.570	1214.556	1134.198	1134.977
5	1234.525	1234.525	1170.988	1170.988

Coupling Parameters

k_L = -.0323; k_C = -.1367 k_L = -.4166 k_C=.0743

Perturbation in Zeroth Cell

Frequency Shifts - π-Mode

δg/g	f	f	f	f
0.000	1234.52497	1234.52497	1170.98800	1170.98800
-0.001	1234.53144	1234.53144	1170.98988	1170.98988
-0.005	1234.55749	1234.55742	1170.99742	1170.99745
-0.010	1234.59043	1234.59015	1171.00686	1171.00698

GAP FIELD SHIFTS

Cell No.	F.C. $\delta I/\lvert I\rvert$	BEVLSQ $\delta I/\lvert I\rvert$	F.C. $\delta I/\lvert I\rvert$	BEVLSQ $\delta I/\lvert I\rvert$
		$\delta g/g = -0.010$		
0	.011335	.010571	.001669	.001643
1	-.004702	-.004685	-.000611	-.000710
2	.000216	.000312	-.000001	.000047
3	.003010	.002811	.000445	.000426
4	-.004957	-.004685	-.000716	-.000710
5	.005610	.005310	.000806	.000805
		$\delta g/g = -0.001$		
0	.001127	.001049	.000169	.000163
1	-.000470	-.000465	-.000063	-.000070
2	.000026	.000031	.000002	.000005
3	.000295	.000279	.000042	.000042
4	-.000489	-.000465	-.000070	-.000070
5	.000554	.000527	.000079	-.000080

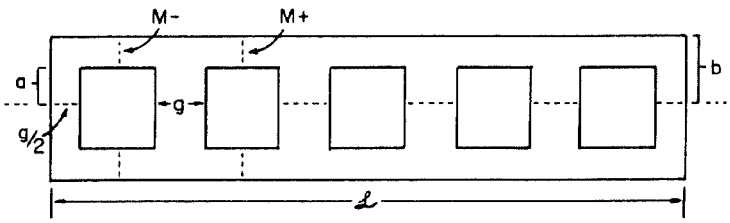

Fig. 1. Geometry of 5-cell azimuthally symmetric structure which is a figure of revolution about the horizontally dashed line. The solid lines are metal surfaces.

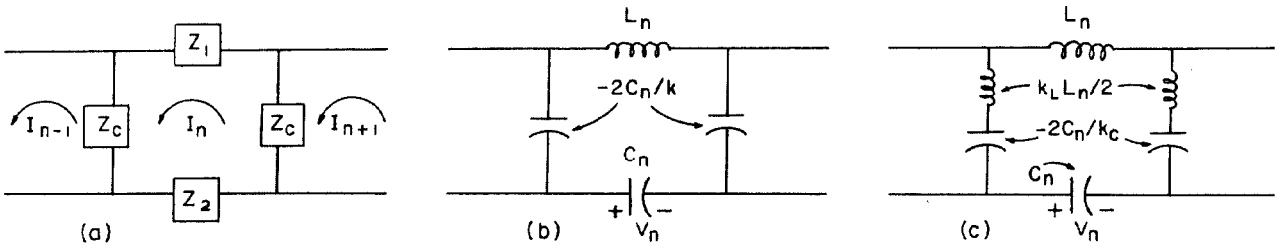

Fig. 2. (a) General circuit; (b) Circuit without ω dependence in the coupling element; (c) Circuit with ω dependence in the coupling element.

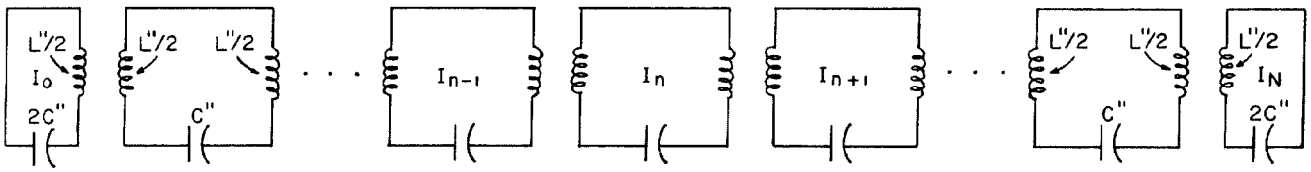

Fig. 3. Half-cell terminated inductively coupled circuit.

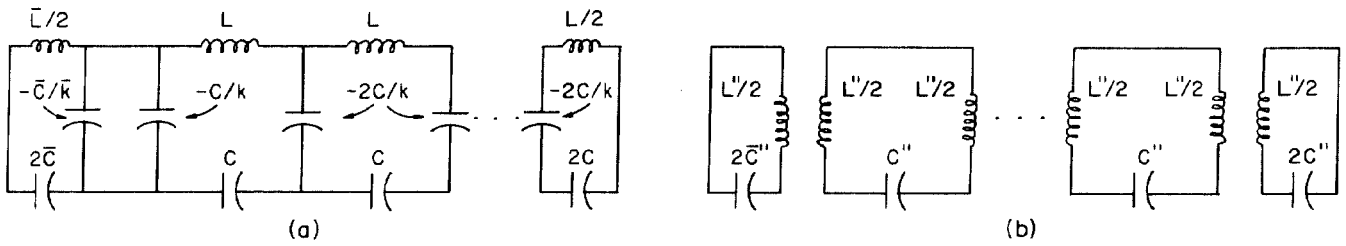

Fig. 4. (a) Perturbation in zeroth cell of Bevensee circuit with constant coupling element; (b) Perturbation in zeroth cell of phenomenological circuit.

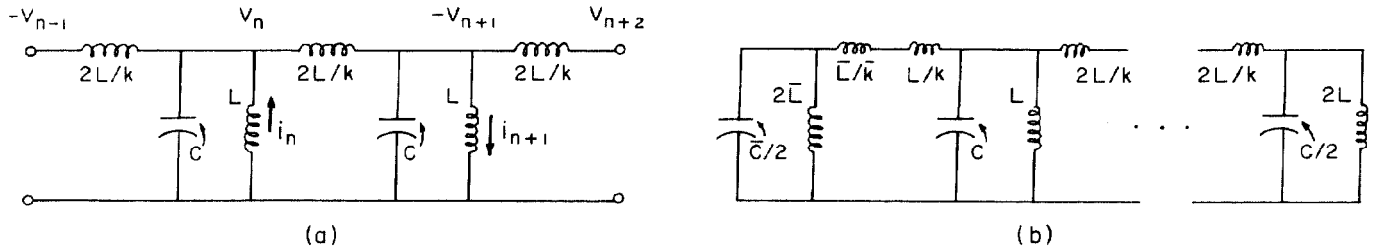

Fig. 5. (a) Equivalent circuit for an even field π-mode; (b) Perturbation in zeroth cell of π-mode circuit.