GENERAL DESIGN EQUATIONS FOR ISOCHRONOUS CYCLOTRONS

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Summary

In a multiparticle variable energy cyclotron there exists a unique sector shape which will provide optimum focussing and isochronism for the full range of particles to be accelerated. To generate this shape, general hard edge equations for the orbit properties are presented. The equations for the betatron frequencies hold for the case when i) the entry and exit spiral angles are unequal ii) the hill and valley fields vary with the radius and iii) for separated sectored cyclotrons also, with the valley field equal to zero. A series of mc^2 type spiral sectored electron cyclotron magnets have been constructed and the hard edge equations have been evaluated against orbit integration results in the measured fields.

I. Introduction

In the hard edge approximation, the equilibrium orbits consist of circular arcs of radii of curvature ρ_H and ρ_V corresponding to the hill and valley fields B_H and B_V as shown in Fig.1. ρ_o is the radius of the circle passing through the points of intersection of the equilibrium orbits with the sectors. When the entry and exit spiral angles ε_1 and ε_2 at radius ρ_o are equal, there must exist a gradient in the hill angle η_o given by

$$\frac{d\eta_o}{d\rho_o} - \frac{1}{\rho_o}(\tan\varepsilon_2 - \tan\varepsilon_1) = 0 \qquad (1)$$

In the hard edge approximation, the average field over the circle of radius ρ_o is

$$B_o = N[B_H\eta_o + B_V\xi_o]/2\pi \qquad (2)$$

In the case when η_o, B_H and B_V are all functions of the radius, the total average field index μ_T' is obtained by differentiating Eq.2 i.e.,

$$\mu_T' = \frac{\rho_o dB_o}{B_o d\rho_o} = (\tan\varepsilon_2 - \tan\varepsilon_1)/R_4 + \frac{N\eta_o\mu_H}{2\pi} + \left[1 - \frac{N\eta_o}{2\pi}\right]\mu_V \qquad (3)$$

$$R_4 = \eta_o[1 + 2\pi B_V/\eta_o N(B_H - B_V)]$$

where we have made use of Eqs. 1 and 2 and introduced the partial hill and valley field indices

$$\mu_H = \frac{\rho_o \partial B_H}{B_o \partial\rho_o} \quad \text{and} \quad \mu_V = \frac{\rho_o \partial B_V}{B_o \partial\rho_o} \qquad (4a,b)$$

A comparison of μ_T' obtained using the hard edge Eq.3 and μ' obtained by a Fourier analysis in the measured field of an N=3, $E=\frac{1}{2}mc^2$ electron cyclotron magnet is shown in Fig.2.

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II. General Design Equations for Orbit Properties

To obtain the general expressions for the betatron frequencies ν_z and ν_r in the hard edge approximation, when ε_1, ε_2, η_o, B_H and B_V are functions of the radius, we may begin with the analytical expressions for the betatron frequencies in terms of the magnetic field coefficients (e.g. Smith and Garren[1])

$$\nu_z^2 = -\mu' + F^2 + \sum_1^\infty \frac{a_n'^2 + b_n'^2}{n^2} + \left(\frac{1-3\mu'}{2}\right)\sum_1^\infty \frac{a_n'^2 + b_n'^2}{n^4} + \frac{1}{4}[2 - d/dx - d^2/dx^2]\sum_1^\infty \frac{a_n'^2 + b_n'}{n^2} \qquad (5)$$

$$\nu_r^2 = 1 + \mu' + \frac{3}{2}\sum_1^\infty \frac{n^2}{(n^2-1)(n^2-4)}(a_n^2 + b_n^2) + \qquad (6)$$

μ' is already given by Eq.3. To obtain the radial derivatives of the Fourier coefficients a' and b' in the hard edge approximation, we Fourier analyse a step wave at radii ρ_o and $\rho_o + \Delta\rho_o$. With reference to Fig.3, the sectors will be displaced by $\Delta\theta_1$ and $\Delta\theta_2$ due to the spiralling at radius $\rho_o + \Delta\rho_o$ (shown by the dotted wave) thus

$$p_n(\rho_o + \Delta\rho_o) - p_n(\rho_o) = \frac{N}{\pi}(B_H - B_V).\,[\Delta\theta_2 \cos n(\beta_o + \eta_o) - \Delta\theta_1 \cos n\beta_o]$$

where p_n is the cosine coefficient of the nth Fourier harmonic. Obtaining the corresponding expression for $q_n(\rho_o + \Delta\rho_o) - q_n(\rho_o)$ and remembering that

$$\tan\varepsilon \equiv \rho_o \frac{d\eta_o}{d\rho_o} \quad \text{and} \quad a_n' \simeq \frac{\rho_o dp_n}{B_o d\rho_o} \qquad (7a, b)$$

and taking limits $\Delta\rho_o \rightarrow 0$

$$a_N'^2 + b_N'^2 = \frac{N^2}{\pi^2}\frac{[B_H - B_V]^2}{B_o^2}[\tan^2\varepsilon_1 + \tan^2\varepsilon_2 - 2\cos N\eta_o \tan\varepsilon_1 \tan\varepsilon_2] \qquad (8)$$

Substituting from Eqs.3 and 8 in Eq.5 and retaining only the first three terms, gives the general hard edge expression for ν_z

$$\nu_z^2 = -(\tan\varepsilon_2 - \tan\varepsilon_1)/R_4 - N\eta_o\mu_H/2\pi - \left[1 - \frac{N\eta_o}{2\pi}\right]\mu_V + F^2 + \frac{1}{\pi^2}\frac{[B_H - B_V]^2}{B_o^2}[S_1\tan^2\varepsilon_1 + S_2\tan^2\varepsilon_2 - 2S_3\tan\varepsilon_1\tan\varepsilon_2] \qquad (9)$$

where $S_1 = S_2 = \sum_1^\infty 1/K^2$ and $S_3 = \sum_1^\infty \frac{1}{K^2}\cos KN\eta_o$

The flutter F^2 and the time averaged field $\bar{B}$ over the orbit are given by

$$F^2 = \frac{(B_H - \bar{B})(\bar{B} - B_V)}{\bar{B}^2}, \quad \bar{B} = \left[\frac{N\eta}{2\pi}\left(\frac{1}{B_H} - \frac{1}{B_V}\right) + \frac{1}{B_V}\right]^{-1}$$

From Eqs. 3 and 6, the corresponding expression for ν_r becomes

$$\nu_r^2 = 1 + (\tan\varepsilon_2 - \tan\varepsilon_1)/R_4 + \frac{N\eta_o}{2\pi}\mu_H + \left[1 - \frac{N\eta_o}{2\pi}\right]\mu_V \quad (10)$$

omitting the third and other terms, which are only significant, because of the $1/N^2$ dependence, when N is low.

Each equilibrium orbit in Fig.1 correspond to a value of γ given by

$$\gamma(\rho_o) = (1+p^2)^{\frac{1}{2}} = \left[1 + \left(\frac{\rho_o \sin\frac{1}{2}\eta_o B_H e}{\sin\frac{1}{2}\eta m_o c^2}\right)^2\right]^{\frac{1}{2}} \quad (11)$$

where η, the turning angle in the hill sector is related to η_o by

$$\cot\tfrac{1}{2}\eta = \cot\pi/N + B_V.(\cot\tfrac{1}{2}\eta_o - \cot\pi/N)/B_H \quad (12)$$

If τ be the time period in an orbit corresponding to γ at ρ_o and τ_1 and γ_1 corresponding values for a reference orbit (e.g the first orbit), then we have

$$\frac{\tau}{\tau_1} = \frac{\gamma[\eta(B_V - B_H) + \frac{2\pi}{N} B_H]}{\gamma_1[\eta_1(B_V - B_H) + \frac{2\pi}{N} B_H]} \quad (13)$$

For isochronism, we require that the ratio τ/τ_1 be constant with the radius.

The Eqs. 9, IO and 13 are valid for homogeneous field sectors and give the orbit properties directly in terms of the sector geometry. Eqs. 9 and 10 are valid for inhomogeneous field sector also. Since in this case, ρ_H and ρ_V will vary with the orbit angle (Fig.1) Eq. 13 may be used only as a first approximation if B_H and B_V do not vary appreciably over the equilibrium orbit. For a separated sectored cyclotron, Eqs. 9, 10 and a modified form of Eq. 13 may be used with $B_V=0$.

III. Scheme for arriving at the Optimum Sector shapes

The sectors alone cannot be made to provide the isochronous field for all the particles. They may however, be shaped to provide the isochronous field (i.e. τ/τ_1 = constant) for a' reference particle' lying in between the extreme cases such that the load on the trim coil currents is a minimum. Due to the different isochronous guiding fields under operating conditions, the $(\nu_r\nu_z)$ curves for each particle will be different on the $(\nu_r\nu_z)$ graph. This family of tunes for all the particles of interest must be kept away from a resonance region. Thus an optimum $(\nu_r\nu_z)$ curve exists for the reference particle which will keep the tunes of all other particles within the operating region on the $(\nu_r\nu_z)$ graph. Thus the optimum sector shapes must provide for the reference particle, simultaneously, i) a specific $(\nu_r\nu_z)$ curve on the $(\nu_r\nu_z)$ graph, and ii) τ/τ_1 = constant with the radius.

Case of homogeneous field sectors

In the case of homogeneous field sectors, the optimum values of the parameters N, B_H and B_V must be fixed. Also, $\mu_H=\mu_V=0$. Since B_H and B_V are constant with the radius an increase in the average field must be obtained by 'flaring' the sectors. The rate of flaring $d\eta_o/d\rho_o$ can be obtained through Eqs.11, 12 and 13 for the condition τ/τ_1= constant and connected to the spiral angles ε_1, ε_2 through Eq.1.

Further, assuming circular equilibrium orbits, we have approximately,

$$\nu_r^2 = \gamma^2 = 1 + \rho_o^2/(\rho_c^2 - \rho_o^2) \quad (14)$$

where ρ_c= cyclotron radius for the reference particle. Eq. 14 can be used in conjunction with the required $(\nu_r\nu_z)$ curve for the reference particle to obtain the radial dependence of ν_z, $\nu_z(\rho_o)$. The Eqs. 1 and 9 may now be solved simultaneously at several radii to obtain the $\varepsilon_1(\rho_o)$ and $\varepsilon_2(\rho_o)$ which will provide the $\nu_z(\rho_o)$ and τ/τ_1 = constant for the reference particle. If this is done, and making use of Eq. 7 , the optimum sector contours are given by

$$\theta_1(\rho_o) = \int \frac{1}{2\rho_o}\left[\frac{R_1^2 - 4R_2}{2R_3(S_1 - S_3)}\right]^{\frac{1}{2}} d\rho_o - \tfrac{1}{2}\eta_o(\rho_o) + \theta_o \quad (15)$$

$$\theta_2(\rho_o) = \theta_1(\rho_o) + \eta_o(\rho_o) \quad (16)$$

where $R_1 = \rho_o d\eta_o/d\rho_o$, $R_2 = (S_1 R_3 R_1^2 + F^2 - R_1/R_4 - \nu_z^2)$

$$R_3 = \frac{1}{\pi^2}\left[\frac{B_H - B_V}{B_o}\right]^2$$

In practice, $d\theta_1/d\rho_o$ may be calculated at a few radii, fitted to a polynomial in radius and integrated.

Case of inhomogeneous field sectors

In this case, the parameters B_H and B_V may also vary with the radius and $\mu_H \neq 0$, $\mu_V \neq 0$. Thus the choice of the variables is not restricted to ε_1 and ε_2 only as in the case of the homogeneous field sectors. A scheme similar to that discussed for the homogeneous field case may be followed, depending upon the particular choice of the variables used.

Separated Sectored Cyclotrons

The above methods may be used, with $B_V=0$, $\mu_V=0$.

IV. Comparison of the hard edge equations with orbit integration results

A series of mc^2 type spiral sectored electron cyclotron magnets were constructed in order to study the validity of the hard edge equations 9, 10 and 13 against orbit integration results for a wide range of parameters. The following three cases were considered: i) A three sectored, $E=\frac{1}{2}mc^2$ electron cyclotron magnet, ii) an eight sectored $E=\frac{1}{2}mc^2$ electron cyclotron magnet and iii) an N=8, separated sectored cyclotron configuration.

Typical hard edge sector parameters obtained after machining and assembly on the pole pieces are shown in Table I. Magnetic field measurements were made in the median plane on a polar grid $B(r,\theta)$. B_H and B_V refer to the measured maxima and minima at each radius. μ_H, μ_V and μ_T' have been calculated using Eqs. 4a, 4b and 3 respectively.

The expected hard edge values of ν_z, ν_r and τ/τ_1 for the three cases were obtained by substituting the sector parameters (e.g Table I) in Eq. 9, 10 and 13 respectively. The corresponding equilibrium orbit properties ν_z', ν_r' and τ'/τ_1' in the measured fields were computed with the equilibrium orbit code, ORBIT. A comparison of the hard edge ν_z and the corresponding orbit integration ν_z' is made for the three cases in Fig. 4a, b and c. A typical comparison of τ/τ_1 and τ'/τ_1' is made for N=3 in Fig. 4d. ν_r and ν_r' are compared for N=8, and for N=8 (separated sector) in Figs. 4e and f.

The hard edge values expected from Eqs. 9, 10 and 13 agree reasonably with the orbit integration results. Thus, Eqs.15 and 16 may be used to obtain the preliminary optimum sector shapes. Once a scale model and orbit integration results become available, the exact optimum shape can be obtained by a differential corrective procedure using a modified Newton-Rhapson method of successive approximations using the hard edge equations 9, 10 and 13.

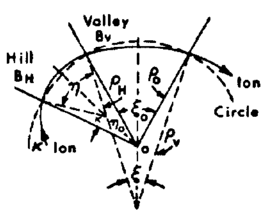

Fig. 1. Section of an equilibrium orbit in the hard edge approximation.

For homogeneous field sectors, an alternative expression for ν_z when $\varepsilon_1 \neq \varepsilon_2$ has also been derived using the 'impulse approximation' approach of Richardson. For the special case of homogeneous field separated sectored cyclotrons, expressions corresponding to Eqs. 9, 10 and 13 derived by G. Schatz, using the matrix method, have been compared with the present set. These and other details will be available elsewhere[2].

References

1. L. Smith and A.A. Garren, UCRL-8598(1959)
2. Orbit Dynamics and Design Theory for Optimization of Sector shapes in Isochronous Cyclotrons. A.Jain and A.S.Divatia, to be published.

Table I

ρ_0	η_0	ε_1	ε_2	B_H	B_V	μ_H	μ_V	μ_T'
cm	...°	...°	...°	G	G	-	-	-
N=3								
7.0	26.5	30.3	42.5	252.9	79.9	.01	-.03	.24
12.0	51.9	36.2	69.5	256.5	84.1	.09	.32	1.10
N=8								
8.0	12.2	35.9	42.2	257.8	94.5	.09	-.05	.37
11.0	17.8	45.9	57.6	256.3	105	-.05	.83	1.10
N=8 Separated Sector								
8.0	11.8	34.1	40.2	274.5	90.5	.86	-.20	.37
11.0	16.5	44.6	55.0	287.4	97.3	.19	.60	1.09

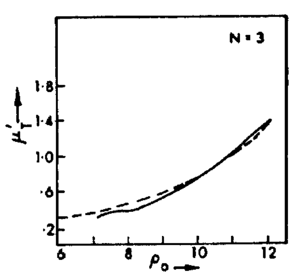

Fig. 2. Comparison of μ_T' expected using Eq.3 (dashed curve) with that obtained by a Fourier analysis in the measured field.

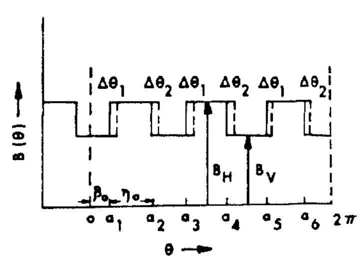

Fig. 3. Step wave approximation B(θ).

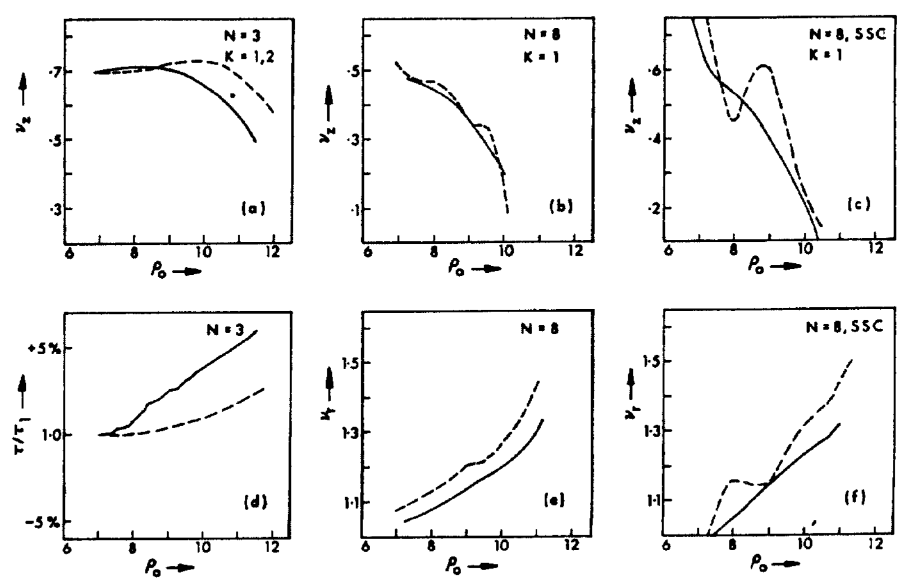

Fig. 4. Variation of the orbit properties with the radius of the mc^2 type electron cyclotron magnets. The dashed curves indicate values expected if the hard edge equations are used. The solid curves represent corresponding values obtained by orbit integration in the measured fields.