EVALUATION OF SYNCHROTRON RADIATION INTEGRALS*

R. H. Helm, M. J. Lee, and P. L. Morton

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

M. Sands

University of California, Santa Cruz, California 95060

Introduction

Many of the important properties of the stored beam in an electron storage ring are determined by integrals,[1] taken around the whole ring, of various characteristic functions of the guide field. Some of the integrals are handled easily, but a few are usually estimated graphically — particularly for alternating-gradient guide fields. This report describes a convenient method for evaluating numerically these recalcitrant integrals.

In the usual linear approximation, the integrals we wish to consider are most conveniently expressed in terms of four (somewhat redundant) functions of the azimuthal coordinates: $\rho(s)$ the radius of curvature of the design orbit, n the field index, $\beta(s)$ the radial betatron function and $\eta(s)$ the off-energy (or "dispersion") function.[2]

The Integrals

We restrict our attention to guide fields made up of a number of magnetic segments — magnets or straight sections. The functions ρ and n are assumed to have constant values within a given magnet, but vary abruptly at the entrance and exit boundaries. The integrals of interest are given by:

$$I_1 = \oint (\eta/\rho)ds = \sum_i \frac{\ell_i}{\rho_i} \langle\eta\rangle_i \tag{1}$$

$$I_2 = \oint (1/\rho^2)ds = \sum_i \frac{\ell_i}{\rho_i^2} \tag{2}$$

$$I_3 = \oint |1/\rho|^3 ds = \sum_i \frac{\ell_i}{|\rho_i|^3} \tag{3}$$

$$I_4 = \oint \frac{(1-2n)\eta}{\rho^3} ds = \sum_i \left[\frac{\ell_i}{\rho_i^3}\langle\eta\rangle_i - 2\ell_i\left\langle\frac{n\eta}{\rho^3}\right\rangle_i\right] \tag{4}$$

$$I_5 = \oint \frac{H}{|\rho|^3} ds = \sum_i \frac{\ell_i}{|\rho_i|^3}\langle H\rangle_i \tag{5}$$

We have used the notation $\langle f\rangle_i$ for the mean value of f in the i^{th} segment whose length is ℓ_i.[1] The function H(s) is defined by

$$H = \frac{1}{\beta}\left\{\eta^2 + (\beta\eta' - \tfrac{1}{2}\beta'\eta)^2\right\} \tag{6}$$

with $\beta' = d\beta/ds$, and $\eta' = d\eta/ds$. It should be noted right away that at least one factor of $1/\rho$ appears in each integral; so the straight sections or pure quadrupoles make no contribution.

The integrals I_2 and I_3 are, evidently, simple sums. Our purpose is to show that the factors $\langle\eta\rangle_i$, $\langle n\eta/\rho^3\rangle_i$, and $\langle H\rangle_i$ that appear in the remaining integrals can be expressed as relatively simple algebraic expressions involving the values of β and η together with their derivatives only at the segment boundaries.

The Beam Parameters

The various performance parameters of storage rings that can be expressed in terms of these integrals are as follows.[1]

1. The dilation factor α, also known as the "momentum compaction," is $\alpha = I_1/L$ where L is the length of the design orbit.

2. The energy loss U_0 in one revolution from synchrotron radiation is

$$U_0 = \left[\frac{2}{3} r_e E_0^4/(mc^2)^3\right] I_2$$

where E_0 is the nominal energy of the stored electrons, r_e is the classical electron radius, and mc^2 is the electron rest energy.

3. The damping of radial betatron oscillation and of energy oscillations are proportional to the damping partition factors J_x and J_ϵ. In terms of our integrals:

$$J_x = 1 - \frac{I_4}{I_2}; \qquad J_\epsilon = 2 + \frac{I_4}{I_2}.$$

Alternatively, we may write the exponential damping coefficients α_x, α_y and α_ϵ as

$$\alpha_x = \frac{r_e}{3}\left(\frac{E_0}{mc^2}\right)^3 \frac{c}{L}(I_2 - I_4) \tag{7}$$

$$\alpha_y = \frac{r_e}{3}\left(\frac{E_0}{mc^2}\right)^3 \frac{c}{L} I_2 \tag{8}$$

$$\alpha_\epsilon = \frac{r_e}{3}\left(\frac{E_0}{mc^2}\right)^3 \frac{c}{L}(2I_2 + I_4) \tag{9}$$

where c is the velocity of light.

4. The distribution of energies induced by quantum emission in a stored beam is — under stationary conditions — characterized by the root-mean-square energy spread σ_ϵ. We may write

$$\left(\frac{\sigma_\epsilon}{E_0}\right)^2 = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc}\left(\frac{E_0}{mc^2}\right)^2 \frac{I_3}{2I_2 + I_4} \tag{10}$$

where $\hbar/mc$ is the reduced Compton wavelength.

5. The quantum excited radial betatron oscillations will, under stationary conditions, have a local root-mean-square displacement $\sigma_{x\beta}(s)$ given by

$$\frac{\sigma_{x\beta}^2(s)}{\beta(s)} = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc}\left(\frac{E_0}{mc^2}\right)^2 \frac{I_5}{I_2 - I_4} \tag{11}$$

Normal Boundary Magnet

We consider now the evaluation of $\langle\eta\rangle$, $\langle n\eta/\rho^3\rangle$ and $\langle H\rangle$ for a particular magnet of length ℓ. For this section we assume that the fringe field boundaries are normal to s, and that within the magnet ρ and n are constant. Under these assumptions the values of η and β inside the magnet may be expressed in terms of the values of these functions and their derivatives at the magnet entrance:

$$\eta = \eta_0 C + \eta_0' \frac{S}{k} + \frac{1}{\rho k^2}(1 - C) \tag{12}$$

$$\beta = \beta_0 C^2 - 2\alpha_0 \frac{CS}{k} + \gamma_0 \frac{S^2}{k^2} \tag{13}$$

where $k^2 = (1-n)/\rho^2$, $C = \cos ks$, $S = \sin ks$, and s is the dis-

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tance from the entrance edge of the magnet. The quantity k^2 is the "restoring force" constant of the particle oscillations. A magnet is focusing if $k^2 > 0$ and $k = (1-n)^{1/2}/\rho$, and is defocusing if $k^2 < 0$ and $k = i(n-1)^{1/2}/\rho$.

The value of $\langle\eta\rangle$ can be found by integrating Eq. (12) directly which yields

$$\langle\eta(\eta_0,\eta_0')\rangle = \eta_0 \frac{\sin k\ell}{k\ell} + \eta_0' \frac{1-\cos k\ell}{k^2\ell} + \frac{1}{\rho}\,\frac{k\ell - \sin k\ell}{k^3\ell} \tag{14}$$

For a normal boundary magnet the variation of n in the fringe field boundary does not contribute to the value of $\langle n\eta/\rho^3\rangle$ (see Appendix). Thus

$$\left\langle\frac{n\eta}{\rho^3}\right\rangle = \frac{n}{\rho^3}\langle\eta\rangle \tag{15}$$

To find the value of $\langle H\rangle$ first we rewrite Eq. (6), the definition of H, in a more convenient form:

$$H = \gamma\eta^2 + 2\alpha\eta\eta' + \beta\eta'^2 \tag{16}$$

where α, γ and η' are given by[2]

$$\alpha = -\frac{1}{2}\beta' = \beta_0 kCS + \alpha_0(C^2 - S^2) - \gamma_0\frac{CS}{k} \tag{17}$$

$$\gamma = \frac{1}{\beta}(1+\alpha^2) = \beta_0 k^2S^2 + 2\alpha_0 kCS + \gamma_0 C^2 \tag{18}$$

$$\eta' = -\eta_0 kS + \eta_0' C + \frac{S}{\rho k} \tag{19}$$

These expressions, together with Eqs. (12) and (13) for η and β, can now be substituted into Eq. (16) for H to give a form that can be straight-forwardly integrated. After some manipulations, the result becomes:

$$\begin{aligned}\left\langle H(\eta_0,\eta_0',\beta_0,\beta_0')\right\rangle &= \gamma_0\eta_0^2 + 2\alpha_0\eta_0\eta_0' + \beta_0\eta_0'^2 \\ &+ \frac{2\ell}{\rho}\left\{-(\gamma_0\eta_0+\alpha_0\eta_0')\frac{k\ell-\sin k\ell}{k^3\ell^2} + (\alpha_0\eta_0+\beta_0\eta_0')\frac{1-\cos k\ell}{k^2\ell^2}\right\} \\ &+ \frac{\ell^2}{\rho^2}\left\{\gamma_0\frac{3k\ell - 4\sin k\ell + \sin k\ell\cos k\ell}{2k^5\ell^3} - \alpha_0\frac{(1-\cos k\ell)^2}{k^4\ell^3} + \beta_0\frac{k\ell - \cos k\ell\sin k\ell}{2k^3\ell^3}\right\}\end{aligned} \tag{20}$$

Non-Normal Boundary Magnet

If the magnet boundaries are not normal to the direction of the design orbit, the above results are modified by the local gradients seen by a particle in passing through the fringe field at an angle (see Appendix):

$$\langle\eta\rangle = \langle\eta(\eta_0,\eta_1')\rangle \tag{21}$$

$$\left\langle\frac{n\eta}{\rho^3}\right\rangle = \frac{n}{\rho^3}\langle\eta(\eta_0,\eta_1')\rangle + \frac{1}{2\ell\rho^2}(\eta_0\tan\phi_1 + \eta_2\tan\phi_2) \tag{22}$$

$$\langle H\rangle = \left\langle H(\eta_0,\eta_1',\beta_0,\beta_1')\right\rangle \tag{23}$$

where

$$\eta_1' = \eta_0' + \frac{\eta_0}{\rho}\tan\phi_1;$$

$$\eta_2 = \eta_0\cos k\ell + \eta_1'\frac{\sin k\ell}{k} + \frac{1}{\rho k^2}(1-\cos k\ell);$$

$$\beta_1' = \beta_0' + 2\frac{\beta_0}{\rho}\tan\phi_1.$$

The boundary rotation at the magnet entrance is ϕ_1, and at the exit ϕ_2. Positive ϕ means radial defocusing at either entrance or exit of the magnet.

References

1. M. Sands, "The physics of electron storage rings. An introduction," Proceedings of the International School of Physics "Enrico Fermi," Course XLVI, ed. B. Touschek (Academic Press, 1971); also SLAC Report 121.

2. E. D. Courant and H. S. Snyder, Ann. of Phys 3, 1 (1958).

APPENDIX

Calculation of Effects of Non-Normal Boundaries

The local gradients which arise from the non-normal boundaries perturb the slopes of the η and β functions and also contribute to the $\langle n\eta/\rho^3\rangle$ integral through its explicit dependence of n(s). To calculate these effects, consider an entrance boundary and assume that the fringe field varies from $B = 0$ to $B = B_0$ in a very short distance, 2ϵ; i.e., $B(s_1 - \epsilon) = 0$, $B(s_1 + \epsilon) = B_0$. The gradient index associated with the boundary rotation ϕ is approximated by

$$n(s) = -\frac{\rho}{B}\left(\frac{\partial B}{\partial \rho}\right) \simeq \frac{\rho}{B}\frac{dB}{ds}\tan\phi_1 \tag{A-1}$$

In the familiar impulse approximation for edge focusing, we assume that η and β are unchanged in going through the fringe field. The changes in η' and β' are

$$\eta_1' = \eta_0' + (\eta_0/\rho_0)\tan\phi_1 \tag{A-2}$$

$$\beta_1' = \beta_0' + 2(\beta_0/\rho_0)\tan\phi_1 \tag{A-3}$$

where ρ_0 is the bending radius in the interior of the magnet.

The increment of the integral $\langle n\eta/\rho^3\rangle$ in the same impulse approximation is

$$\delta\langle n\eta/\rho^3\rangle = \frac{\eta_0\tan\phi_1}{\ell(B\rho)^2}\int_{-\epsilon}^{\epsilon} B\frac{dB}{ds}\,ds = \frac{\eta_0\tan\phi_1}{2\ell\rho_0^2} \tag{A-4}$$

We employ the usual convention for the signs of the entrance and exit boundary angles; (positive ϕ means radial defocusing at either entrance or exit). See Fig. A-1. The results for the exit boundary are completely analogous.

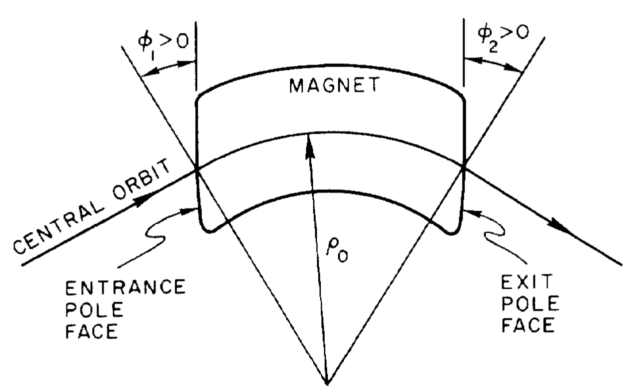

FIG. A-1--Field Boundaries for a bending magnet.