EFFECT OF RANDOM FLUCTUATIONS ON SYNCHROTRON PHASE MOTION

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Summary

Previous treatment of this subject has been given by Hereward and Johnsen.[1] In this note a unified presentation is made of the growth of longitudinal phase area occupied by the beam due to random fluctuations of the magnetic field, radio frequency, and cavity voltage. The description is characterized by a linear treatment of the synchrotron motion, introduction of an envelope function similar to that used in betatron motion, and the use of Nyquist's theorem to obtain power spectra. Application is made to the NAL booster.

Linearized Unperturbed Phase Motion

If one uses canonically conjugate variables (P,η) to express the longitudinal motion of a particle relative to the synchronous or reference particle, one has in the linear approximation

$$\dot{P} = -\frac{V}{2\pi h}\cos\phi_R\eta \qquad \dot{\eta} = \frac{h^2\omega_R^2\kappa_R}{E_R}P. \tag{1}$$

The peak voltage per turn is V, ϕ_R is the reference phase, h the harmonic number, ω_R the angular frequency of the reference particle, E_R the total energy of the reference particle, and

$$\kappa_R = \frac{\gamma_T^2 - \gamma_R^2}{\gamma_T^2\left(\gamma_R^2-1\right)}. \tag{2}$$

Here γ_R is the kinematical gamma of the reference particle and γ_T is the transition gamma.

In the following it is useful to change the independent variable from time to s using[2]

$$s = \int_0^t \frac{h^2\omega_R^2\kappa_R}{E_R}\,dt. \qquad [(\text{eV-sec})^{-1}] \tag{3}$$

The motional equations become

$$P' = -K\eta \qquad \eta' = P, \tag{4}$$

where

$$K = \frac{E_R V}{2\pi h^3\omega_R^2\kappa_R}\cos\phi_R. \qquad [(\text{eV-sec})^2] \tag{5}$$

and prime indicates differentiation with respect to s. In this form one sees by analogy with betatron motion that the solution of Eq. (4) may be put in the form

$$\eta = \sqrt{2\beta W}\,\sin\left(\int_0^s \frac{ds}{\beta} + \gamma\right) \tag{6}$$

where β is the amplitude function satisfying

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + \beta^2 K = 1.$$

Since the constants of the motion (W,γ) are canonically conjugate one has for the phase area

$$E = \iint dP\,d\eta = \iint dW\,d\gamma = 2\pi W, \tag{8}$$

where the last equality is true if W is interpreted as the boundary curve of a group of particles

$$W = \frac{1}{2}\beta P^2 + \alpha P\eta + \frac{1}{2}\gamma\eta^2. \tag{9}$$

Here

$$\alpha = -\frac{1}{2}\beta' \qquad \gamma = \frac{1}{\beta}(1+\alpha^2). \tag{10}$$

Perturbed Phase Motion

If the reference motion is perturbed such that $V\to V+\Delta V$, $\omega_R\to\omega_R+\Delta\omega_R$, and $\omega_{RF}\to\omega_{RF}+\Delta\omega_{RF}$ where ω_{RF} is the angular frequency of the rf, then Eq. (4) becomes[3]

$$J' = -K\eta + F \qquad \eta' = J + G, \tag{11}$$

where

$$F = \frac{E_R\Delta V\sin\phi_R}{2\pi h^3\omega_R^2\kappa_R}$$
$$G = \frac{E_R}{h^2\omega_R^2\kappa_R}\left(\frac{h\omega_R}{\gamma_T^2}\frac{\Delta B}{B} + \Delta\omega_{RF}\right). \tag{12}$$

It will be noted that $\Delta\omega_R$ has been expressed in terms of ΔB the perturbation in the magnetic field. The units are $(\text{eV-sec})^2$ for F and eV-sec for G.

Our only concern will be with the change in the invariant W due to the perturbations. Thus using the solution of Eq. (11) substituted into Eq. (9) one has

$$W = \frac{1}{2}(X_1+H_1)^2 + \frac{1}{2}(X_2+H_2)^2, \tag{13}$$

where

$$X_1 = \eta_1(0)P(0) - P_1(0)\eta(0)$$
$$X_2 = \eta_2(0)P(0) - P_2(0)\eta(0), \tag{14}$$

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and

$$H_1 = \int_0^s \eta_1 F ds - \int_0^s P_1 G ds$$
$$H_2 = \int_0^s \eta_2 F ds - \int_0^s P_2 G ds. \tag{15}$$

Here (P_1, η_1) and (P_2, η_2) are two independent solutions of Eq. (4) such that $\eta_2 P_1 - \eta_1 P_2 = 1$.

In terms of these independent solutions the general solution of Eq. (7) that represents the motion of a group of particles matched to the small amplitude bucket shape is

$$\beta = \eta_1^2 + \eta_2^2. \tag{16}$$

From Eq. (10) one then finds

$$\alpha = -(\eta_1 P_1 + \eta_2 P_2) \qquad \gamma = P_1^2 + P_2^2. \tag{17}$$

Statistical Treatment of Fluctuations

Chandrashekar[4] shows that a density distribution $\rho(W,t)$ in which the variable W is governed by a random walk process obeys the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial W}\left[-D_1\rho + \frac{1}{2}\frac{\partial}{\partial W}(D_2\rho)\right] \tag{18}$$

where

$$D_1 = \frac{d}{dt}\left\langle \Delta W\right\rangle_{Av} \qquad D_2 = \frac{d}{dt}\left\langle (\Delta W)^2\right\rangle_{Av} \tag{19}$$

where the brackets represent ensemble averages.

By associating ΔW with $W(s) - W(0)$ and invoking the ergodic theorem which asserts that ensemble averages and time averages are identical, Eqs. (13) and (19) give for Eq. (18):

$$\frac{\partial \rho}{\partial t} = \left[\frac{1}{2t}\left(H_1^2+H_2^2\right)\right] \cdot \frac{\partial}{\partial W}\left(W\frac{\partial \rho}{\partial W}\right). \tag{20}$$

In this form one sees that a new independent variable is useful. Namely,

$$w = \frac{1}{2}\int_0^t \frac{1}{t}\left(H_1^2+H_2^2\right) dt. \tag{21}$$

Then Eq. (20) becomes

$$\frac{\partial \rho}{\partial w} = \frac{\partial}{\partial W}\left(W\frac{\partial \rho}{\partial W}\right) \tag{22}$$

whose fundamental solution is

$$\rho = \frac{1}{w} e^{-\frac{W}{w}}. \tag{23}$$

from which w is seen to be the average value of W over the distribution.

If the individual contributions to the fluctuations are independent, one may consider each one in turn and add. If, on the other hand, feedback is employed to correlate $\Delta\omega_R$ with $\Delta\omega_{RF}$, the problem is more complex and is not considered here. The contributions to $H_1^2 + H_2^2$ are

$$\left(H_1^2+H_2^2\right)_{cav} = \pi \int_0^t \frac{\sin^2\phi_R}{4\pi^2 h^2}\,\beta\, J_{cav}(\Omega)dt, \tag{24}$$

$$\left(H_1^2+H_2^2\right)_{Mag} = \pi \int_0^t \frac{h^2\omega_R^2}{\gamma_T^4 B^2}\,\gamma\, J_{Mag}(\Omega)dt, \tag{25}$$

$$\left(H_1^2+H_2^2\right)_{RF} = \pi \int_0^t \gamma\, J_{RF}(\Omega)dt \tag{26}$$

where by Nyquist's theorem the spectral densities are

$$J_{cav}(\Omega) = \frac{2}{\pi} kT\, R_{cav}(\Omega) \tag{27}$$

$$J_{Mag}(\Omega) = \frac{2}{\pi} kT\, \left|T_B(\Omega)\right|^2 \text{Real } Z_{Mag}(\Omega) \tag{28}$$

$$J_{RF}(\Omega) = \frac{2}{\pi} kT\, \left|T_{RF}(\Omega)\right|^2 \text{Real } Z_{RF}(\Omega) \tag{29}$$

and Ω is the synchrotron frequency

$$\Omega = \frac{\dot{s}}{\beta}. \tag{30}$$

The other quantities are Boltzmann's constant k, the absolute temperature for the unit under consideration T, the transfer function between magnetic field and volts across magnet T_B in G/V, the transfer function between the angular rf frequency and the low level rf voltage T_{RF} in 2πHz/V. The impedances in Ohms are: R_{cav} for the shunt impedance of rf cavities around the ring, Z_{mag} the total magnet impedance, and Z_{RF} the impedance of the low level unit driving the low level frequency changing circuit.

Results for NAL Booster

Only the random fluctuation $\Delta\omega_{RF}$ is significant in producing a growth in the longitudinal phase space area associated with the beam. The function $\gamma(t)$ characterizing the beam bunch shape is peaked at transition. Equation (21) gives the average value of $W = E/2\pi$ to be expected at any time during the cycle due to random fluctuations. Using $kT = 5 \times 10^{-21}$J, $T_{RF}(0) = 2\pi \times 7.5$ MHz/V and $Z_{RF}(0) = 1$ MΩ one finds at transition

$$\Delta E = .006 \text{ eV-sec (one bunch)}. \tag{31}$$

This is to be compared with an initial beam area of

$$E = .02 \text{ eV-sec (one bunch)}. \tag{32}$$

References

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4. S. Chandrashekar, Plasma Physics, The University of Chicago Press, Third Impression 1965, Notes compiled by S.K. Trehan, p. 183.

5. F. Reif, Fundamentals of Statistical and Thermal Physics, McGraw Hill Book Co., New York 1965, p. 587.