STOCHASTICITY LIMIT AND TURBULENT MOTION OF PHASE-SPACE FLUID

L. C. Teng
National Accelerator Laboratory*
Batavia, Illinois

Introduction

This is an attempt to provide an alternative picture for visualizing the motion of a particle under the influence of non-linear forces. This picture leads to a condition for the onset of stochastic motion (stochasticity limit) somewhat different from that given by Chirikov.[1]

It is well known that for particles performing a Hamiltonian motion, the ensemble of their phase-points behaves like an incompressible fluid in the phase-space. Although the details of the motion of this phase-fluid in the presence of non-linear forces are rather complex, certain gross features are clear from the analytical and numerical studies made to date. The motion near the origin (small amplitude) is dominated by the linear forces and has the characteristics of smooth laminar rotation about the origin. As one goes away from the origin the non-linear forces play an increasingly more important role, until beyond some radial distance regions of stochastic motion appear which have very much the characteristics of turbulence. In fact, the concept of stochasticity has exactly the same geometrico-statistical content as that of turbulence. In a physical fluid both turbulent and laminar flows are governed by the same Navier-Stokes equation. The only distinction is that when the turbulence regime as specified by the Reynolds condition is reached the vorticity becomes suddenly very large and the velocity correlation length drops precipitously to a very small value so that the flow looks irregular and choppy. In the same manner both stochastic and regular motions of the particles are governed by the same Hamiltonian with the only distinction that in the stochasticity domain the motion looks irregular and choppy, hence the motion of the phase-fluid looks turbulent.

The program is now clear: if we can cast the motion of the phase-fluid given by the canonical equations of a Hamiltonian in the form of the Navier-Stokes equation for a viscous and incompressible fluid we should expect that the Reynolds condition for the onset of turbulence will give the stochasticity limit. This is manifestly true since the Reynolds condition is a purely geometrico-statistical condition based on a mathematical similarity parameter and, as mentioned before, the transition to stochasticity, as well as to turbulence, is only a mathematical correlation phenomenon with no change in physical content. This program will bring the mathematics into a form for which the statistical correlation properties have been properly parametrized and thoroughly investigated. We will treat here only the case of one-dimensional motion. The generalization to more than one dimension is straightforward but can be expected to be complicated.

*Operated by Universities Research Association Inc. under contract with the United States Atomic Energy Commission.

The Hamiltonian

We shall write the Hamiltonian as

$$H(q,p;s) = \frac{1}{2}(p^2+Kq^2) - F \tag{1}$$

where (q,p) are the conjugate canonical variables, s is the independent variable (length along the equilibrium orbit), $K = K(s)$ is the periodic linear "force coefficient," and $F = F(q,p;s)$ contains all non-linear terms.

It is well known that the linear motion can be transformed to a harmonic oscillation by the Floquet transformation. This consists of a canonical transformation to the action-angle variables (J,ϕ)

$$\begin{cases} q = \sqrt{2J}\,\sqrt{\beta}\,\cos\phi \\ p = \sqrt{2J}\,\frac{1}{\sqrt{\beta}}\,(\sin\phi - \alpha\cos\phi) \end{cases} \tag{2}$$

where $\alpha(s)$ and $\beta(s)$ are the Courant-Snyder linear oscillation parameters,[2] and a transformation of the independent variable to $\theta = \int ds/\beta$. The resulting Hamiltonian is

$$H(J,\phi;\theta) = J - \beta F. \tag{3}$$

So defined θ advances 2π per oscillation. Considering $r \equiv \sqrt{2J}$ and ϕ as the polar coordinates of the phase plane and using the complex vector $z \equiv \sqrt{2J}\,e^{i\phi} = re^{i\phi}$ we get from the canonical equations the complex velocity field of the phase-fluid

$$\begin{aligned} v = \frac{dz}{d\theta} &= \left(\frac{1}{\sqrt{2J}}\frac{dJ}{d\theta} + i\sqrt{2J}\,\frac{d\phi}{d\theta}\right)e^{i\phi} \\ &= \left(\frac{1}{\sqrt{2J}}\frac{\partial H}{\partial\phi} - i\sqrt{2J}\,\frac{\partial H}{\partial J}\right)e^{i\phi} = -2i\frac{\partial H}{\partial z^*}. \end{aligned} \tag{4}$$

Navier-Stokes Equation

We assume the entire phase-plane to be filled with a phase-fluid of uniform density unity. (Since the phase-fluid is incompressible the density is an invariant.) The Navier-Stokes equation of motion for such a fluid is then

$$\frac{\partial\vec{v}}{\partial\theta} + \vec{v}\cdot\vec{\nabla}\vec{v} = -\vec{\nabla}P + \vec{\nabla}\cdot\left[\eta\,(\vec{\nabla}\vec{v})_{\text{sym}}\right] \tag{5}$$

where the left side is the transport derivative of $\vec{v}$ (the transport acceleration) and the right side is the force per unit volume, $P = P(r,\phi;\theta)$ is the pressure field, η is the viscosity assumed to be a function of position and $(\vec{\nabla}\vec{v})_{sym}$ is the symmetrized rate-of-strain tensor. For the 2-dimensional fluid here it is convenient to use the complex vector notation. In terms of complex vectors this equation becomes

$$\frac{\partial v}{\partial \theta} + \left(v\frac{\partial}{\partial z} + v^*\frac{\partial}{\partial z^*}\right)v = -2\frac{\partial P}{\partial z^*} + 2\frac{\partial}{\partial z}\left(\eta\frac{\partial v}{\partial z^*}\right). \quad (6)$$

Our job is to find P and η in terms of the Hamiltonian H such that the motion of the phase-fluid given by the canonical equation (4) is a solution of Eq. (6). Substituting Eq. (4) in the left side of Eq. (6) we get

$$\begin{aligned}\frac{\partial v}{\partial \theta} &+ \left(v\frac{\partial}{\partial z} + v^*\frac{\partial}{\partial z^*}\right)v \\ &= \frac{\partial}{\partial \theta}\left(-2i\frac{\partial H}{\partial z^*}\right) - 2i\left(\frac{\partial H}{\partial z^*}\frac{\partial}{\partial z} - \frac{\partial H}{\partial z}\frac{\partial}{\partial z^*}\right)v \\ &= -2\frac{\partial}{\partial z^*}\left(i\frac{\partial H}{\partial \theta}\right) - 2i\frac{\partial}{\partial z^*}\left(H\frac{\partial v}{\partial z}\right) + 2i\frac{\partial}{\partial z}\left(H\frac{\partial v}{\partial z^*}\right) \\ &= -2\frac{\partial}{\partial z^*}\left(i\frac{\partial H}{\partial \theta} + 2H\frac{\partial^2 H}{\partial z\partial z^*}\right) + 2\frac{\partial}{\partial z}\left(iH\frac{\partial v}{\partial z^*}\right).\end{aligned}$$

Equating this to the right side of Eq. (6) we can identify

$$\left\{\begin{aligned}P &= i\frac{\partial H}{\partial \theta} + 2H\frac{\partial^2 H}{\partial z\partial z^*} \\ \eta &= iH\end{aligned}\right. \quad (7)$$

For a given Hamiltonian H with pressure P and viscosity η defined by Eq. (7) the Navier-Stokes equation (6) will describe properly the motion of the phase-fluid. Eq. (7) has two unpleasant features.

(1) Both P and η contain imaginary terms. A purely imaginary term implies an antisymmetric tensor. This is a little unusual but since we are not dealing with a physical fluid there is nothing basically objectionable. Moreover, if the viscosity must be a tensor, a 2-dimensional antisymmetric tensor having only one parameter is almost as simple as a scalar.

(2) If the Hamiltonian is an explicit function of θ so will be η. In this case we should first perform a series of Moser transformations[3] to eliminate the explicit θ-dependence of the Hamiltonian. This is rather impractical. Here, we will assume the nonlinear forces to be weak. Then, following the spirit of the first-order perturbation approximation we can average the small nonlinear perturbation term βF in the Hamiltonian [Eq. (3)] over the unperturbed motion. This means putting $\phi = \psi-\theta$ in βF and averaging over θ. Since η is a property value and not a dynamical variable the θ-average should be taken over a range of 2π instead of over the entire unperturbed motion. Denoting this average by $\langle\ \rangle$ we can rewrite the viscosity as

$$\eta = i\langle H\rangle = i(J - \langle\beta F\rangle). \quad (8)$$

Reynolds Condition

For unit density the local Reynolds number should be a length multiplied by a scaling velocity divided by the local viscosity. The most natural choice of the length is the length of z and that of the scaling velocity is the velocity of the linear motion

$$v_o \equiv -2i\frac{\partial J}{\partial z^*} = -iz.$$

Also, since the viscosity is purely imaginary (antisymmetric tensor) we should form a purely imaginary combination out of z and v_o. All these considerations lead to a Reynolds number Re defined as

$$Re \equiv \frac{zv_o^* - z^*v_o}{\eta} = \frac{4}{1-\frac{1}{J}\langle\beta F\rangle}. \quad (9)$$

Let us now state that the motion of the phase-fluid is turbulent where $Re \to \infty$ (or $Re < 0$). This gives the "stochasticity limit," as

$$\frac{1}{J}\langle\beta F\rangle > 1. \quad (10)$$

This condition looks quite curious. Since the viscosity is, aside from an i, just the Hamiltonian this condition amounts simply to the statement that when the Hamiltonian is zero the viscosity is zero, hence the Reynolds number is infinity and the flow is turbulent. The linear part of the Hamiltonian is proportional to J and, hence, increases as the square of the radial distance from the origin. In regions at some larger radius the increasingly dominant non-linear part can turn the Hamiltonian negative and make the flow turbulent. This condition is a great relaxation from the very restrictive sufficient-but-not-necessary condition that the motion is stochastic if the curvature of the surface $H = H(J,\phi)$ is everywhere negative.

Example

Take a specific simple $F = \sum_{k=0}^{\infty} a_k q^k$. Then

$$\beta F = \sum_k a_k \beta^{\frac{k}{2}+1}(2J)^{\frac{k}{2}}\cos^k\phi. \quad (11)$$

We shall assume that a_k, same as β, is also periodic in θ with period $2\pi\nu$ where ν is the usual betatron oscillation wave number. We can then write

$$a_k\beta^{\frac{k}{2}+1} = \sum_{m=-\infty}^{\infty} b_{k,m} e^{i\frac{m}{\nu}\theta}, \quad \left(b_{k,-m} = b_{k,m}^*\right). \quad (12)$$

Also expanding

$$\cos^k\phi = \sum_{n=-k}^{k} g_{k,n} e^{in\phi} \qquad \text{where}$$

$$g_{k,n} = g_{k,-n} = \text{real and non-negative}, \quad (13)$$

we have

$$\beta F = \sum_k (2J)^{\frac{k}{2}} \sum_n g_{k,n} e^{in\phi} \sum_m b_{k,m} e^{i\frac{m}{\nu}\theta}. \quad (14)$$

To average over the linear motion we put $\phi = \psi-\theta$ then average over θ. This gives

$$\langle\beta F\rangle = \sum_k (2J)^{\frac{k}{2}} \sum_n g_{k,n} e^{in\psi} \sum_m b_{k,m} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\left(\frac{m}{\nu}-n\right)\theta} d\theta \right]$$

$$= \sum_k (2J)^{\frac{k}{2}} \sum_n g_{k,n} e^{in\psi} \sum_m b_{k,m} \left[\frac{\sin\left(\frac{m}{\nu}-n\right)\pi}{\left(\frac{m}{\nu}-n\right)\pi} \right]. \quad (15)$$

The expression in the bracket considered as a function of m has a peak of unity at $m = n\nu$ and falls off symmetrically to the first zeros at $m = n\nu \pm \nu$. We can define

$$B_{k,n} \equiv \sum_m b_{k,m} \left[\frac{\sin\left(\frac{m}{\nu}-n\right)\pi}{\left(\frac{m}{\nu}-n\right)\pi} \right] \equiv |B_{k,n}| e^{i\alpha_{k,n}} \quad (16)$$

and write the "stochasticity limit" as

$$\frac{1}{J} \langle\beta F\rangle = 4\sum_k r^{k-2} \sum_{n=0}^{k} g_{k,n} |B_{k,n}| \cos(n\psi+\alpha_{k,n}) > 1. \quad (17)$$

The maximum value of $\frac{1}{J}\langle\beta F\rangle$ at a given radius $r \equiv \sqrt{2J}$ is when all $\cos(n\psi+\alpha_{k,n}) = 1$. If we are interested only in the radius beyond which the motion can be stochastic we can write

$$\left[\frac{1}{J}\langle\beta F\rangle\right]_{max} = 4\sum_k r^{k-2} \sum_{n=0}^{k} g_{k,n} |B_{k,n}| > 1. \quad (18)$$

If, further, $b_{k,m}$ does not vary much over the range of m values from $n\nu-\nu$ to $n\nu+\nu$ we can write approximately $B_{k,n} \cong \nu b_{k,\overline{n\nu}}$ where $\overline{n\nu}$ denotes the integer nearest to $n\nu$, and for the stochasticity limit

$$4\nu \sum_k \sum_{n=0}^{k} g_{kn} |b_{k,\overline{n\nu}}| r^{k-2} > 1. \quad (19)$$

For this example and under the same approximation Chirikov's prescription gives

$$\sum_{n,m} (\Delta\nu_{n,m}) = 4\nu \sum_n \sum_{k=n}^{\infty} k g_{k,n} \; r^{k-2} \sum_m |b_{k,m}|$$

$$= 4\nu \sum_n \sum_{k=n}^{\infty} n k g_{k,n} |b_{k,\overline{n\nu}}| r^{k-2} > 1. \quad (20)$$

The summation domains in Eqs. (19) and (20) are identical. The only difference is the presence of the factor nk in the Chirikov condition which accentuates the effect of high orders in non-linearity and resonance.

Discussion

Several steps in the development of the theory are open to question. These and some other critical comments are summarized below.

Once the starting philosophy is accepted the development of the theory is rather unique and straightforward. The fact that the phase motion can be described by the Navier-Stokes equation with the pressure field and the viscosity simply related to the Hamiltonian is, in itself, interesting.

In comparison, the derivation and interpretation of Chirikov's condition are less transparent. Furthermore, Chirikov's prescription is based on resonances with periodic forces. The transition from regular to stochastic motion should not depend qualitatively on the periodicity of the forces. The theory developed here is independent of the strength and the structure of the forces.

We assumed the critical Reynolds number to be infinity whereas any universal constant value is allowed. The appropriate value may have to be determined empirically. Also, because of the arbitrariness in sign of the upstairs term in the Reynolds number it may well be that the condition should be written as $|Re| > critical$. In that case the turbulent regions can be multiply connected.

The weakest point in the development is the approximation of eliminating the θ-dependence of η by averaging it over one oscillation of the linear motion. On the other hand, the proper procedure of making Moser transformations is difficult in practice. The validity of this approximation as well as the difference between the condition (19) and the Chirikov condition (20) can presumably be resolved by computer experiments.

The extension to multidimensional motion is complicated but straightforward. The pressure field P and the viscosity η are equally simply related to the Hamiltonian through the unit antisymmetric (symplectic) tensor. Again, difficulties arise in eliminating the s-dependence of η.

Mathematically, the behavior of turbulence in the phase-fluid is analogous to that in a real physical fluid. Therefore, the cascade from large to small eddies, the similarity hypotheses, and the consequent spectral laws for the structural functions as formulated by Kolmogorov[4] should apply equally well to turbulence in the phase-fluid. This application may yield valuable insight into the statistical characteristics of the stochastic motion of particles under the influence of non-linear forces.

The author would like to acknowledge many helpful discussions with Dr. D. A. Edwards during the course of this work.

References

1. B.V. Chirikov, Report 267, Nucl. Phys. Inst. Siberian Sec. USSR Acad. of Sci. Novosibirsk (1969). Trans. by A.T. Sanders, CERN Trans. 71-40 (1971).

2. E.D. Courant and H.S. Snyder, Ann. of Phys. 3, 1-48 (1958).

3. J. Moser, Nachr. Akad. Wiss. Göttingen IIa, No. 6, 87 (1955).

4. A.N. Kolmogorov, Dokl. Akad. Nauk, SSSR 30, No. 4, 299-303; 31, No. 6, 538-541; 32, No. 1, 16-18 (1941).