DESIGN OF DIPOLE MAGNET WITH CIRCULAR IRON SHIELD

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Summary

Complex variable function theory is used to formulate the magnetostatic problem of a rectangular current block of uniform current density located within a circular region bounded by infinite permeability iron. The magnetic field from the superposition of several blocks arranged to exhibit dipole symmetry is calculated using a generalization of Cauchy's integral formula. Subsequent multipole expansion yields expressions for the multipole coefficients which through the use of variable metric minimization permits block arrangements to be found for rather pure dipole fields. Application is made to superconducting dipoles for both a pancake arrangement and a shell arrangement of the individual blocks.

Magnetostatics in Two Dimensions with Complex Variables

If one uses the complex variable H to represent the two rectangular components of the magnetic field

$$H = H_x + iH_y, \tag{1}$$

and, in addition, if one defines

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{2}$$

then the equations of magnetostatics may be written conveniently as

$$\frac{\partial H^*}{\partial z^*} = -2\pi i J, \tag{3}$$

where J is the current density.

The integral relation corresponding to Eq. (3) is provided by a generalization of Cauchy's formula.[1] Thus

$$f(s) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-s}\,dz - \frac{1}{\pi}\iint \frac{\partial f}{\partial z^*}\cdot\frac{1}{z-s}\,dxdy. \tag{4}$$

In our application C may be taken as the curve at infinity where one may confidently expect that reasonable functions can be set equal to zero. Hence, if H* is identified with f one has

$$H^*(s) = -\frac{1}{\pi}\iint \frac{\partial H^*}{\partial z^*}\cdot\frac{1}{z-s}\,dxdy. \tag{5}$$

By using Eq. (3) this becomes

$$H^*(s) = 2i\iint \frac{J}{z-s}\,dxdy. \tag{6}$$

If the current density is different from zero in several isolated regions one has

$$H^*(s) = 2i\sum_k \iint \frac{J_k}{z_k-s}\,dx_k dy_k. \tag{7}$$

Dipole Field within a Circular Iron Boundary

Dipole symmetry requires that four current regions be introduced as shown in Fig. 1. Thus

$$z_1 = z \qquad z_2 = -z \qquad z_3 = z^* \qquad z_4 = -z^* \tag{8}$$

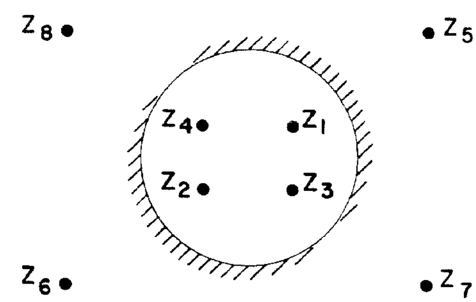

Fig. 1 Arrangement of Current Filaments for Dipole Symmetry

The boundary conditions on the curve $|z| = b$ can be met by locating image positions at

$$z_5 = \frac{b^2}{z^*} \qquad z_6 = -\frac{b^2}{z^*} \qquad z_7 = \frac{b^2}{z} \qquad z_8 = -\frac{b^2}{z} \tag{9}$$

Dipole symmetry and image currents require

$$J_k dx_k dy_k = -(-1)^k J dxdy. \tag{10}$$

Equation (7) then becomes

$$H^*(s) = 4i\iint J dxdy \left\{ \frac{z}{z^2-s^2} + \frac{z^*}{z^{*2}-s^2} + \frac{\frac{b^2}{z}}{\left(\frac{b^2}{z}\right)^2 - s^2} + \frac{\frac{b^2}{z^*}}{\left(\frac{b^2}{z^*}\right)^2 - s^2} \right\} \tag{11}$$

The integration is only over current regions in the first quadrant.

If the current density is constant over any region then Beth's theorem[2] may be utilized to simplify further the contribution from that region. Beth states that for a

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function $w(z) = 1/(z-s)$

$$\iint w(z)\,dxdy = \frac{1}{2i}\oint_C w(z)(z-s)^*dz \qquad (12)$$

where C is the curve bounding the region. As a corrolary one has

$$\iint w(z^*)\,dxdy = -\frac{1}{2i}\oint_C w(z^*)(z-s^*)dz^* \qquad (13)$$

where C is traversed in the same sense in Eq. (12) and Eq. (13). Thus, for constant current density, Eq. (11) becomes for $|s|$ less than b

$$H^*(s) = 2J\oint_C \left\{\left[\frac{zz^*-ss^*}{z^2-s^2} + \frac{\frac{b^2}{z}z^*}{\left(\frac{b^2}{z}\right)^2 - s^2}\right]dz - \left[\frac{zz^*-ss^*}{z^{*2}-s^2} + \frac{\frac{b^2}{z^*}z}{\left(\frac{b^2}{z^*}\right)^2 - s^2}\right]dz^*\right\} \qquad (14)$$

Multipole Expansion

The currents are located externally to a region which is the useful bore of the magnet. In the region of this bore one may expand Eq. (14) in powers of s. Thus letting

$$\boxed{T_n(z) = \frac{1}{z^n} + \left(\frac{z}{b^2}\right)^n} \qquad (15)$$

one has

$$\boxed{H^*(s) = 4iJ\sum_{n=1,3...} s^{n-1}\, Imag \oint_C T_n(z)z^*dz} \qquad (16)$$

This is the basic formula used in the design procedure that follows. The effect of several blocks is obtained by superposition.

Pancake Model

In this case the current blocks are parallel to the x-axis. If one puts

$$A_n = Imag \oint_C T_n(z)\; z^*dz, \qquad (17)$$

an integral which is evaluated using numerical integration. Equation (16) becomes

$$\boxed{H^*(s) = 4iJ\sum_{n=1,3...} A_n s^{n-1}} \qquad (18)$$

Shell Model

The primary arrangement for the shell model is indicated in Fig. 2 where two blocks touch on the median plane. Along line 1 the

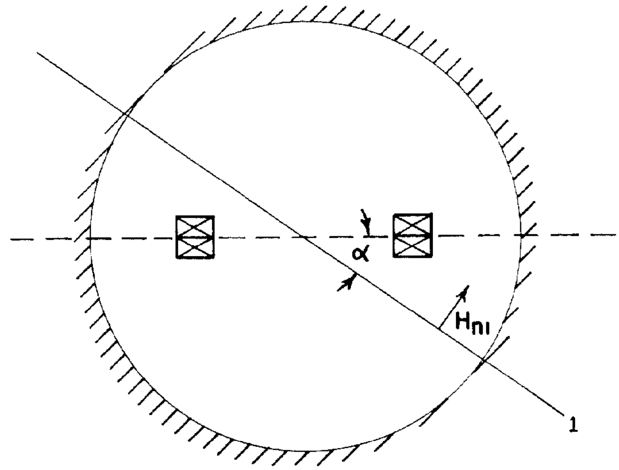

Fig. 2 Block Arrangement for Primary Calculation for Shell Model

field is

$$H_1^* = 4iJ\sum_n A_n r^{n-1}[\cos(n-1)\alpha - i\sin(n-1)\alpha] \qquad (19)$$
$$= H_{1x} - iH_{1y}.$$

The field normal to line 1 is

$$H_{n1} = H_{1x}\sin\alpha + H_{1y}\cos\alpha = -4J\sum_n A_n r^{n-1}\cos n\alpha \qquad (20)$$

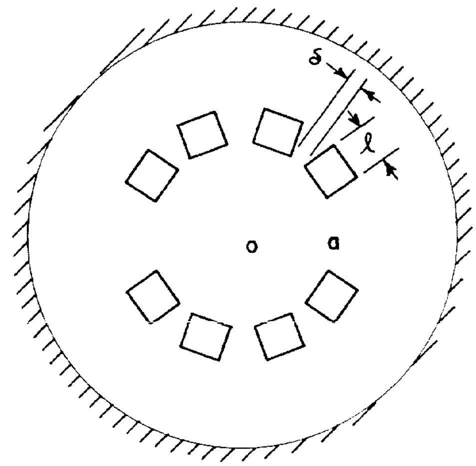

Fig. 3 Block Arrangement in Shell Model

By symmetry the resulting field on the median plane for a dipole configuration of rotated blocks as shown in Fig. 3 is

$$\boxed{H_y = -8J\sum_n A_n x^{n-1}\cos n\alpha} \qquad (21)$$

Superposition of the results from many blocks located at various angles α yields a shell distribution of rectangular wires. Figure 3 gives the geometry for blocks arranged on a circular arc of inner radius a.

From this geometry one finds that if ℓ is the block height and δ the distance between the innermost corners of adjacent blocks

$$2a\sin\frac{\beta}{2} - \ell\cos\frac{\beta}{2} = |\delta|. \qquad (22)$$

For the m-th block

$$\alpha_m = \left(m-\frac{1}{2}\right)\beta \qquad (23)$$

Adjustment of Block Location

One wishes to locate the current blocks in such a manner that a nearly uniform field is produced within the aperture delineated by the blocks. To this end it is desirable to minimize the energy difference between the actual field and a field assumed uniform over the aperture. As an example, consider the pancake model. The energy contained within a circle of radius a is

$$E = \frac{1}{8\pi}\int_0^{2\pi}\int_0^{a} HH^* r\,dr\,d\theta. \qquad (24)$$

Equation (19) gives

$$E = \frac{2}{\pi} J^2 \sum_{nm}\sum A_n A_m \int_0^a r^{n+m-1} dr \int_0^{2\pi} e^{i(n-m)\theta} d\theta \qquad (25)$$

or

$$E = 2J^2 \sum_n \frac{A_n^2}{n} a^{2n}. \qquad (26)$$

Since the term for n = 1 represents the energy content assuming a uniform field, the function to be minimized is

$$\Delta E = 2J^2 \sum_{n=3,5...} \frac{A_n^2}{n} a^{2n}. \qquad (27)$$

The variables in the minimization are the locations of the various blocks which are implicit in Eq. (17) by the assumption of superposition. In our case the minimization was carried out using a variable metric minimization code due to Davidon.[3] The results indicate that block locations for the pancake model and shell configurations for the shell model can be found for which the field uniformity is better than ±.05 percent over the useful aperture.

References

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2. R.A. Beth, An Integral Formula for Two-Dimensional Fields, J. Appl. Phys. 38, 4689 (1967). See also: R.A. Beth, J. Appl. Phys. 40, 4782 (1969).

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