MINIMIZATION OF ABERRATIONS IN BEAM LINE DESIGN*

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Summary

The aberrations present in a beam line represent the departure from the ideal first-order design. Such aberrations may be of second and higher order. Individual second-order terms may be represented by a matrix T.[1] We derive a formalism for representing the net contribution of such terms to the beam dimensions. Corrective elements such as sextupoles or non-linearities in bending magnets may be employed. A method for minimizing the effects of second-order aberrations has been derived. This method has been incorporated into the program TRANSPORT.[2] Higher-order effects may now arise from the coupling of second-order terms. A computer program TURTLE[3] is used to evaluate such effects.

Introduction

Charged particle beam lines are, in general, initially designed to achieve certain first-order characteristics. One might, for example, wish to fix one or more elements of the first-order transfer matrix or place a condition on the phase ellipse at some point in the beam line. Representing a beam line to first order is a reasonable approximation for particle trajectories which are close to the central axis of the beam line and whose momentum is close to that of the central design momentum of the beam line.

It is often desirable to extend the formalism to second order by the use of additional terms. These second-order terms represent a departure from the ideal first-order design. It is therefore desirable to be able to assess their effect, and, if found excessive, to introduce correcting elements which will minimize it.

Such corrections may optimize the beam line design to second order, only to introduce aberrations of third and higher order. These higher-order effects are not revealed with a second-order matrix element approach. Therefore one must employ methods of ray tracing or a reasonable approximation thereof to determine such high-order terms.

First- And Second-Order Transfer Matrices

The position and direction of the trajectory of a charged particle at a given point in a beam line may be represented by a six component vector $X = (x, \theta, y, \phi, \ell, \delta)$, here written in row form. The components x and y represent the transverse location of the particle relative to the central trajectory, while the quantities θ and ϕ represent respectively the angles made with the central trajectory in the same planes. The quantity ℓ is a path length difference between the trajectory of interest and the central trajectory, and δ is the fractional deviation of the momentum from the central design value. The components will also be denoted by appending a subscript to the letter X.

To first order the effect of a beam line may be represented by a square matrix R.[4] The passage of a charged particle through the beam line is given by the equation $X(1) = RX(0)$, which $X(0)$ is the initial and $X(1)$ the final coordinate vector of the particle. The transfer matrix for the entire beam line may be obtained by multiplying together the transfer matrices for the magnets and drift spaces comprising the beam line.

This formalism may be extended to second order by the use of an additional term giving:

$$X_i(1) = \sum_j R_{ij} X_j(0) + \sum_{jk} T_{ijk} X_j(0) X_k(0) \quad (1)$$

where T is the second-order transfer matrix. Just like the first-order matrix, it may be calculated from the R and T matrices of the individual elements in the beam line. The R and T matrices for such individual elements have been calculated extensively by Brown.[1]

Estimates of Beam Dimensions

In accelerator and beam line studies we are often more interested in the behavior of an aggregate of charged particles than in that of a single particle. A multi-dimensional phase ellipse formalism is often used to represent such an aggregate of particles. The particles are assumed to have their coordinate vectors lying in the interior of a six-dimensional ellipsoid whose equation is given as:

$$X^T \sigma^{-1} X = 1 \quad (2)$$

The maximum extents of the envelope of the particles in each dimension are given by the square roots of the diagonal members of the σ matrix. The orientation of the ellipse is determined by the off-diagonal matrix elements.

The σ matrix at the final point in the beam line may be obtained from the initial beam matrix via the equation:

$$\sigma(1) = R\sigma(0) R^T \quad (3)$$

When effects of second- and higher-order are included, the final distribution is no longer given by an ellipsoid. We can however interpret the elements of the σ matrix as giving the second moments of the phase-space distribution. Given such an interpretation we can once again take the square roots of the diagonal elements as giving representative beam dimensions.

To calculate the σ matrix at the final point in the beam line we need not only the R and T matrices and initial σ matrix, but the fourth moments of the initial distribution.[5] It is therefore necessary to be given more details about the initial distribution than can be obtained from the σ matrix.

We choose to work with a model where the initial distribution is a multi-dimensional gaussian. This model has the advantage that the higher moments are easily calculated. It also gives results which, under more careful analysis, prove to be of the right order of magnitude and, if anything, an overestimate. Thus if aberrations are important, their effect will show up in the final σ matrix.

For such an initial distribution centered on the beam line axis we label the fourth moments by the letter σ but with four indices. We then can derive:[5]

$$\sigma_{ijk\ell} = \sigma_{ij}\,\sigma_{k\ell} + \sigma_{ik}\,\sigma_{j\ell} + \sigma_{i\ell}\,\sigma_{jk} \quad (4)$$

The final second moments are given by:

$$\sigma_{ij}(1) = \sum_{k\ell} R_{ik}\,R_{j\ell}\,\sigma_{k\ell}(0) + \sum_{k\ell mn} T_{ik\ell}\,T_{jmn}\,\sigma_{k\ell mn}(0) \quad (5)$$

The centroid is now not centered on the beam axis but has coordinates given by:

$$\overline{X_i}(1) = \sum_{jk} T_{ijk}\,\sigma_{jk}(0) \quad (6)$$

The final distribution will now be centered around the new centroid position, with half widths given by the square roots of the diagonal elements of the matrix of second moments about that centroid. Such second moments $\overline{\sigma}$ are given by:

$$\overline{\sigma}_{ij}(1) = \sum_{k\ell} R_{ik}\,R_{j\ell}\,\sigma_{k\ell}(0) + 2\sum_{\ell m}\left(\sum_k T_{ik\ell}\,\sigma_{km}(0)\right)\left(\sum_n T_{jmn}\,\sigma_{\ell n}(0)\right) \quad (7)$$

For initially off-axis distributions more terms are involved. It is recommended that the reader consult the references. The procedure described has been included in the computer program TRANSPORT.[2] It will therefore calculate the net effect of aberrations to the beam dimensions at any point in the beam line.

Optimization To Second Order

Once one has an estimate of the net effect of beam line aberrations, one can determine whether they constitute a problem. If so, a procedure exists for minimizing them. This procedure has also been incorporated into TRANSPORT.

To correct second-order terms in a beam line, one may employ sextupoles or sextupole components in a bending magnet. The latter may be obtained by curving the entrance or exit faces or tailoring the pole tips to produce a quadratic term in the central field. One can now use TRANSPORT to determine the required strengths of any combination of the above named correcting elements. The program will adjust the strengths of the indicated correcting elements to minimize the effect of any aberrations desired. It can also vary the position of a sextupole within a drift space to determine optimum placement of the corrective element.

In adjusting such corrective elements one can impose any of several possible constraints. One may wish simply that the net effect at a given point of all aberrations be minimized. One then uses TRANSPORT to minimize a certain element of the σ matrix.

Alternatively, only a subset of the second-order terms may constitute a problem. For example, the effect of certain terms may be merely to shift the focal plane. Others may represent imperfect focusing on this new plane. One may wish to minimize the effect of the latter while leaving the former unconstrained. With TRANSPORT one can now constrain any set of T matrix elements directly. One can also assign relative weights to the different T matrix elements to produce the optimal configuration. The program will then determine the solution which best satisfies the indicated constraints.

Higher-Order Terms

Once a design has been optimized to second order, one then wonders about the effect of even higher-order terms. For example, has one introduced higher-order effects through the correcting elements used for second-order optimization?

In some cases, investigating higher orders will require a genuine ray-tracing computer program which follows individual particles through the beam line by integrating the equations of motion. For many applications a lumped element ray-tracing computer program TURTLE[3] will provide much information.

In TURTLE, as in ray-tracing programs, rays are run through one at a time. Here though, a ray is carried across a single beam line element via a transfer matrix. Thus results are truncated to second order only for single elements.

Chromatic effects are treated exactly for quadrupoles and sextupoles, since the transfer matrix for each ray is calculated from the actual momentum of the ray. For bending magnets this is not possible, so chromatic effects are represented via first- and second-order matrix elements. In high energy, separated function beams this proves to be a satisfactory approximation.

Higher-order effects due to the cumulative effect of second-order correcting elements can also be exhibited. Such effects have often proven important in NAL beams.

Several designs, optimized to second order, had to be discarded due to the effect of higher-order terms.

Non-linearities in quadrupoles may also be included. Multipoles up to and including a 40-pole are allowed.

The beam profile at any point may be represented via one- and two-dimensional histograms. Any of the six coordinates cited above may be histogrammed at any location in the beam line. For a two-dimensional histogram any coordinate at any location can be displayed with any other coordinate at any other point.

Slits and magnet apertures may also be included. A flag may then be placed on a histogram so that rays are entered only if they are transmitted to some later point in the beam line. Thus good field regions, acceptances, and transmissivities may be determined.

Below we show some sample output from TURTLE. In Figure 1 is shown a one-dimensional display of an uncorrected horizontal profile at the focus of an NAL beam. In Figure 2 the profile is narrowed by the addition to the beam line of two correcting sextupoles. In Figure 3 two additional sextupoles have been added to similarly reduce second-order terms in the vertical plane. The profile is noticeably wider. This effect is due to the coupling of second-order terms to produce undesirable higher-order effects.

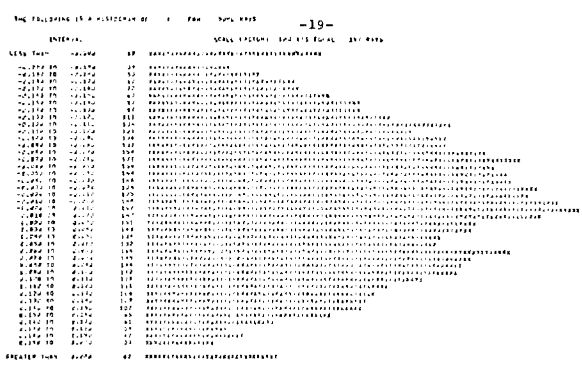

Figure 1: Uncorrected beam profile showing chromatic aberration

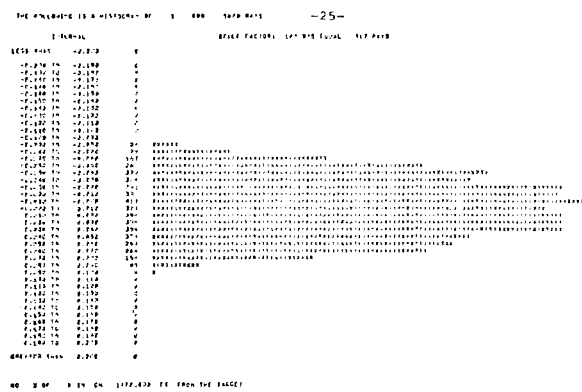

Figure 2: Corrected beam profile using two sextupoles

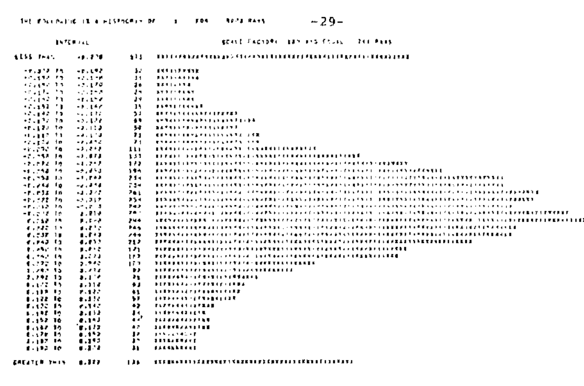

Figure 3: Beam profile with four sextupoles

References

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