SPACE CHARGE EFFECTS IN HIGH CURRENT LINEAR ACCELERATOR TRANSPORT SYSTEMS*

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Summary

The transverse and longitudinal effect in short high current RF buckets of cylindrical geometry and uniform charge distribution are considered in free space. Two cases are examined (a) The beam is radially constructed (magnetically) and (b) The beam is free both radially and longitudinally.

Transverse expansion of a beam in space with a non-symmetrical conducting boundary is presented.

Introduction

In contrast to other writers that solve the problem in the stationary frame and then transfer these results to the laboratory frame ending up with unrealistic beam configurations, we prefer to transfer the electric and magnetic fields of a charge element and solve the problem in the laboratory frame where we can use the beam geometry we deem realistic.

The electric and magnetic fields due to volume charge element $\rho d\varphi$ (where ρ = Ne/volume) may be obtained in the laboratory frame by a Lorentz transformation from the Coulomb field of the stationary volume charge element $\rho d\varphi$ and are given by:[1]

$$\vec{E} = (\rho d\varphi)\,\hat{R}\,\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}} \qquad \hat{v},\ \hat{R} \text{ unit vectors}$$

$$\vec{B} = \beta\hat{v}\times\vec{E} = \frac{\rho d\varphi}{R^2}\,\frac{\beta^2(1-\beta^2)}{(1-\beta^2\sin^2\theta)^{3/2}}$$

and the force element dF acting on an electron at any point in this field is $\vec{dF} = e(\vec{E}+\vec{\beta}v\times\vec{B})$ or

$$\vec{dF} = \frac{e\rho d\varphi}{R^2}\,\frac{(1-\beta^2)}{(1-\beta^2\sin\theta)^{3/2}}\left[(1-\beta^2)\hat{R}-\beta^2(\hat{v}\cdot\hat{R})\hat{v}\right] \quad \text{or}$$

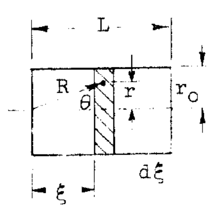
$$\vec{dF} = \frac{e\rho d\varphi}{R^2}\,\frac{(1-\beta^2)\sin\theta}{(1-\beta^2\sin^2\theta)^{3/2}}\,\hat{y} + \frac{e\rho d\varphi}{R^2}\,\frac{(1-\beta^2)\cos\theta}{(1-\beta^2\sin^2\theta)^{3/2}}\,\hat{x}$$

$$\vec{dF} = \frac{e\rho d\varphi}{4\pi\varepsilon_o R^2}\,\frac{(1-\beta^2)\sin\theta}{(1-\beta^2\sin^2\theta)^{3/2}}\,\hat{y} \quad \frac{e\rho d\varphi}{4\pi\varepsilon_o R^2}\,\frac{(1-\beta^2)\cos\theta}{(1-\beta^2\sin^2\theta)^{3/2}}\,\hat{x} \quad (1)$$

Longitudinal space charge effects for a radially restrained beam

The $\vec{dFy}$ component of the force element dF is 0. Equation (1) becomes

$$\vec{dF} = \frac{e\rho d\varphi}{4\pi\varepsilon_o R^2}\,\frac{(1-\beta^2)\cos\theta}{(1-\beta^2\sin^2\theta)^{3/2}}\,\hat{x}$$



Consider a cylindrical beam of volume charge density $\rho = Ne/r_o^2L$.

The longitudinal force F_x on an end electron at point O in the laboratory frame is

$$F_x = +\frac{e}{4\pi\varepsilon_o}\,\frac{\rho}{\gamma^2}\int_\xi\int_r\int_\varphi \frac{\cos\theta\, r dr d\varphi d}{(1-\beta^2\sin^2\theta)^{3/2}R^2}$$

and taking the volume integral

$$F_x = \frac{Ne^2}{2\pi\varepsilon_o\gamma^4 m_o}\,\frac{1}{r_o^2L}\left[\sqrt{(\gamma L+R_o)^2-2R_o\gamma L_o}-(\gamma L_o+R_o)\right]$$

The differential equation of motion is:

$$\frac{d}{dt}\left(m\gamma\frac{dL}{dt}\right) = m_o\gamma^3\,\frac{d^2L}{dt^2} = F_x \quad \text{or}$$

$$\frac{d^2L}{dt^2} = \frac{Ne^2}{2\pi\varepsilon_o\gamma^4 m_o}\,\frac{1}{r_o^2L}\left[\sqrt{(\gamma L_o+R_o)^2-2r\gamma L_o}-(\gamma L_o+r_o)\right]$$

Substituting $x=t\varphi$ and $\varphi = \beta c$ we arrive at the differential equation of motion that was used for our computer program (Fig. 1)

$$\frac{d^2L}{dt^2} = \frac{1}{m_oc^2 2\pi\varepsilon_o\gamma^2(\gamma^2-1)}\,\frac{1}{r_o^2L_o}\left[\sqrt{(\gamma L_o+r_o)^2-2r_o\gamma L_o}-(\gamma L_o-r_o)\right] \quad (3)$$

Since $F_x = F_xe$ the electric field F_x that an end electron will be subjected to is $E_x = F_x/e(4)$ in computer calculations shown on (Fig. 2) based on (4). The energy spread was computed for very small intervals and the change in charge density of the bunch as the packet was expanding longitudinally was taken in consideration.

Transverse Space Charge Effects

When we investigate the transverse effects of $\vec{dF}(1)$ due to symmetry the x component becomes 0 and we have

$$\vec{dF} = \frac{e\rho d\varphi}{4\pi\varepsilon_o R^2}\,\frac{(1-\beta^2)\sin\theta}{(1-\beta^2\sin^2\theta)^{3/2}}\,\hat{y}$$

For a cylindrical beam of length L_o and radius r_o the total radical force Fy on a particle of r distance from the axis is:

$$Fy = \int\int\int_{-\theta_1}^{+\theta_1}\frac{e\rho}{4\pi\varepsilon_o R^2}\,\frac{(1-\beta^2)\sin\theta}{(1-\beta^2\sin^2\theta)^{3/2}}\,r dr d\varphi d\theta$$

and

$$Fy = \frac{e\rho\pi r}{2\pi\varepsilon_o\gamma^2}\left[1-\frac{1}{2}\left(\frac{r_o}{\gamma L}\right)^2+\frac{3}{4}\left(\frac{r_o}{\gamma L}\right)^4-\frac{25}{16}\left(\frac{r_o}{\gamma L}\right)^6+\dots\right]$$

The alternating series converges fast when $\gamma \gg 1$. Now the radial motion of the particle is given by $d^2r/dt^2 = Fg/m_o\gamma$. Here the assumption is made that the charge density is not so high that the peripheral electrons acquire relativistic velocites in the radial direction.

Changing the independent variable from t to x by noting that x = tv and v = βc

$$\frac{d^2r}{dx^2} = \frac{e\rho r_o}{2\varepsilon_o m_o c^2\gamma^2(\gamma^2-1)}\left[1-\frac{1}{2}\left(\frac{r_o}{\gamma L}\right)^2+\left(\frac{r_o}{\gamma L}\right)^4-\dots\right] \quad (5)$$

Equations (3) and (5) solved simultaneously for very small increment of x by α computer program yield the results of the radial and longitudinal expansion of an RF bucket.

Some of these results are shown in Figs. 3,4,5,6.

The Effect of Non-Symmetrical Conducting Boundary on the Transverse Expansion of the Beam.

In examining the transverse effect of space change it is sometimes convenient to use the expression of the continuous current with good accuracy. A hollow circular conductor due to symmetry has no transverse effect and the equation of motion is

$$m_o\gamma \frac{d^2r}{dt^2} = \frac{eIn}{2\pi r}\ \frac{1-\beta^2}{\beta}\ (1) \qquad n = \frac{\mu_o}{\varepsilon_o}$$

However in a bending magnet the beam will be between parallel plates and the magnet poles and the image charges and currents will effect the beam in the transverse direction.

Due to the energy spread the cross section of the beam in the magnetic field will be eliptical and the equation of motion:

$$m_o\gamma \frac{d^2b}{dt^2} = \frac{eIn}{2\pi\beta}\ \frac{2(1-\beta^2)}{(\alpha+b)} + b\left(\frac{\varepsilon_1}{h^2} + \frac{\varepsilon_2}{g^2}\right)$$

2α - beam width, $2b$ - beam height, $2h$ - vacuum chamber height, $2g$ - magnet gap.

The term $b(\varepsilon_1/h^2 + \varepsilon_2/g^2)$ gives the effect of image charges and currents.

Derivation of ε_1 and ε_2. The beam is of eliptical cross section with major axis 2α and minor $2b$ The electric field at a point on the envelope due to a line change $dl = \rho r dr d\varphi$ is $dE = \rho r dr d\varphi/2\pi\varepsilon_o r$ (MKS system) and the field due to the total beam is

$$E = \iint \frac{\rho d\varphi dr}{2\pi\varepsilon_o} = \int_{-\pi/2}^{+\pi/2} \frac{\rho}{2\pi\varepsilon_o}\ r_1 d\varphi$$

from symmetry $E_x = 0$ so

$$E_y = \int_{-\pi/2}^{+\pi/2} \frac{\rho\ r_1}{2\pi\varepsilon_o}\ \cos\varphi\ d\varphi$$

The equation of the beam eliptical envelope in polar coordinates is $\left(\frac{r\cos\varphi-b}{b}\right)^2 + (r\sin\varphi/\alpha)^2 = 1$ and

$$E_y = \int_{-\pi/2}^{+\pi/2} \frac{\rho}{2\pi\varepsilon_o}\ \frac{2\alpha^2 b\cos^2\varphi d\varphi}{\alpha^2\cos^2\varphi + b^2\sin^2\varphi}$$

$$E_y = \frac{2\pi\alpha b\ \rho}{2\pi\varepsilon_o(\alpha+b)} = \frac{\lambda}{2\pi\varepsilon_o}\ \frac{2}{\alpha+b}$$

In the same way the magnetic component

$$B_x = -\frac{\mu_o V\rho}{2\pi} \iint_{\text{eliptic}} d\varphi dr = -\frac{\mu_o V\lambda}{2\pi}\ \frac{2}{\alpha+b}$$

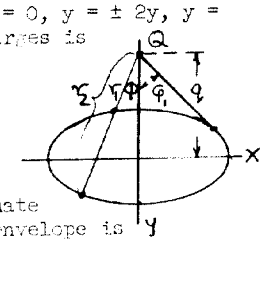
and the force

$$F_y = e\left[E_y + VB_x\right] = \frac{e\lambda}{2\pi\varepsilon_o}\ \frac{2}{\alpha+b}\left[1 - \mu_o\ \varepsilon_o V^2\right]$$

$$F_y = \frac{eIn}{2\pi}\ \frac{1-\beta^2}{\beta}\ \frac{2}{\alpha^2+\beta^2}$$

Considering the effect of the image charges we note that when the beam center is at $x = 0$, $y = 0$ the center of image charges are at $x = 0$, $y = \pm 2y$, $y = \pm 4h$ etc. and E_y due to image charges is

$$E_y = \int_{-\varphi_1}^{+\varphi_1} \frac{\rho\cos\varphi}{2\pi\varepsilon_o}\ (r_2-r_1)d\varphi$$



where Q is the pole of the polar coordinates (r,φ). In this coordinate system the equation of the beam envelope is

$$\left(\frac{r\cos\varphi-b}{b}\right) + \left(\frac{r\sin\varphi}{a}\right)^2 = 1 \text{ gives}$$

$$r_{1,2} = a^2q = \frac{a^2q\cos\varphi \pm \alpha b\sqrt{a^2-(q^2+c^2)\sin^2\varphi}}{(a\cos\varphi)^2 + (b^2\sin\varphi)^2} \qquad c^2 = a^2-b^2$$

substituting r_1 and r_2 in the expression for E_y and integrating for $\varphi = \varphi_1$, $r_1 = r_2$, $\sin\varphi_1 = a/\sqrt{q^2+c^2}$

$$E_y = \frac{\lambda}{\pi\varepsilon_o}\ \frac{1}{\sqrt{q^2+c^2}+q}$$

The image charges above the medium plane are $q = 2nh-b$ and for those below $q = 2nh+b$. They have the opposite sign for odd values of n and the same sign as the beam for even values. The y component then of the field at a point P(ob) due to all the charges is

$$E_y = \frac{\lambda}{\pi\varepsilon_o}\sum(-1)^{n-1}\left[\frac{1}{\sqrt{(2nh-b)^2+c^2}+2nh-b} - \frac{1}{\sqrt{(2nh+b)^2+c^2}+2nh+b}\right]$$

$$F_y = \frac{\lambda b\varepsilon_1}{2\pi\varepsilon_o h^2}$$ plus a term of the order of $(b/2h)^2$ or smaller and

$$\varepsilon_1 = \sum_{n=1}^{\infty}(-1)^n\ \frac{1}{n\ \sqrt{n^2+(c/2h)^2}+n^2+(c/2h)^2}$$

In calculating the effects of the image currents we note the images are now with respect to the pole phases and that all image currents have the same direction as the beam current. Therefore,

$$B_x = \frac{\mu_o I}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{(2ng-b)^2+c^2}+2ng-b} - \frac{1}{\sqrt{(2ng+b)^2+c^2}+2ng+b}\right]$$

or $B_x = \frac{\mu_o I}{2\pi}\ \frac{b}{g^2}\ \varepsilon_o$ where

$$\varepsilon_o = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+(c/2g)^2}+n^2+(c/2g)^2}$$

For a beam of initial radius of $2 = 0.5$ cm $\gamma = 1/\sqrt{1-\beta^2} = 20$ and $a = 2$cm, $b = 0.75$ cm, $h = 1.5$ cm $g = 2$ cm, $\varepsilon_1 = 0.33$ and $\varepsilon_2 = 0.75$ the E_y component of the space charge force is 48 times the force the beam would be subjected if it was traveling inside a circular pipe.

References

1. Landau and Lifshits, "The Classical Theory of Fields", pg. 99.

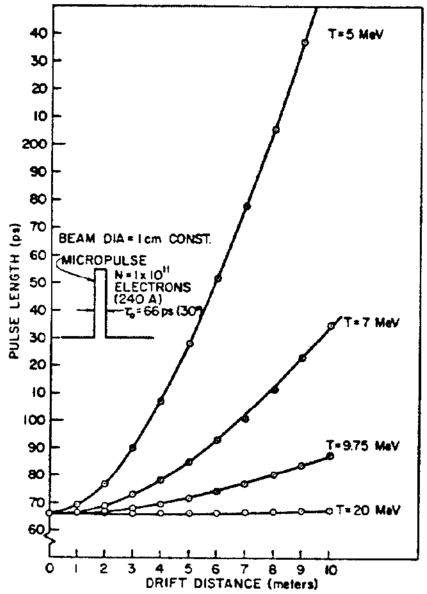


FIG. 1. PULSE LENGTH IS INCREASED BY THE LONGITUDINAL SPACE CHARGE EFFECT; BEAM IS RADIALLY CONSTRAINED TO A DIAMETER OF 1 cm.

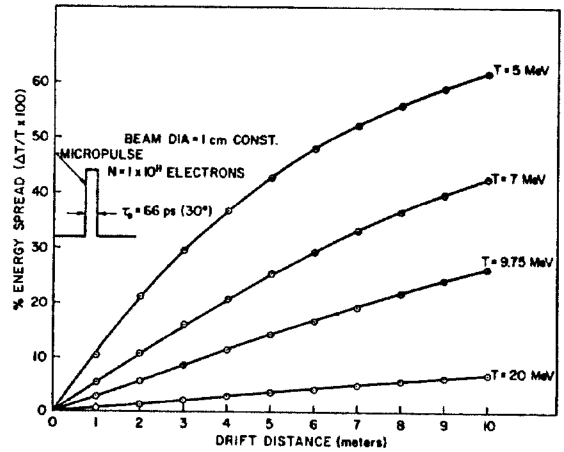


FIG. 2. ENERGY SPREAD DUE TO SPACE CHARGE FOR A RADIALLY CONSTRAINED BEAM. EFFECT IS ENERGY DEPENDENT.

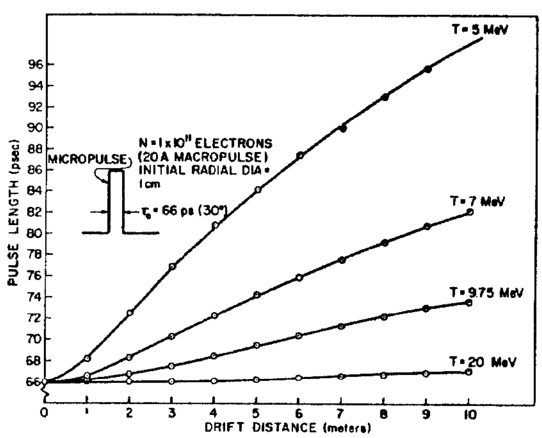


FIG. 3. PULSE LENGTH IS INCREASED BY THE LONGITUDINAL SPACE CHARGE EFFECT; BEAM IS RADIALLY EXPANDING. EFFECT IS ENERGY DEPENDENT.

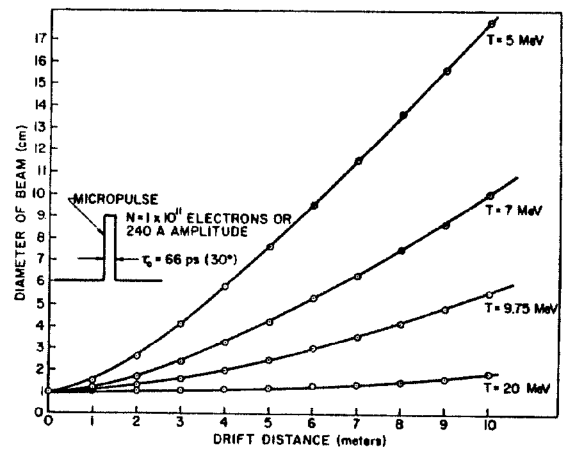


FIG. 4. RADIAL DIMENSION OF THE BEAM IS INCREASED BY THE RADIAL SPACE CHARGE EFFECT.

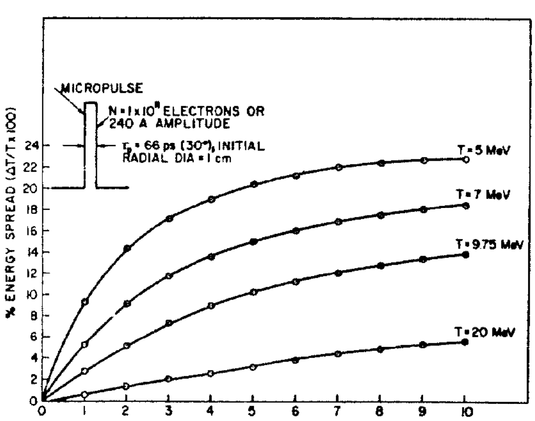


FIG. 5. ENERGY SPREAD INTRODUCED BY THE LONGITUDINAL SPACE CHARGE EFFECT; BEAM IS RADIALLY EXPANDING. EFFECT IS ENERGY DEPENDENT.

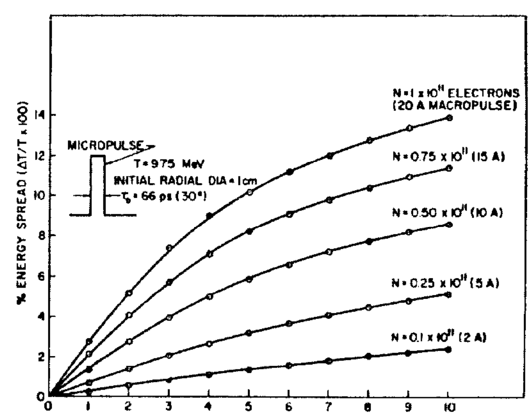


FIG. 6. ENERGY SPREAD INTRODUCED BY THE LONGITUDINAL SPACE CHARGE EFFECT; BEAM IS RADIALLY EXPANDING. EFFECT IS DEPENDENT UPON THE NUMBER OF ELECTRONS IN THE MICROPULSE.