NEW METHODS FOR MULTITURN INJECTION INTO SYNCHROTRONS*†

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Summary

In Part I we derive equations for the matched betatron phase space acceptance polygon for the *n*th turn and for the phase area swept out by the septum. A variable rate of shifting of the equilibrium orbit past the septum is considered with the object being to maximize the accelerated charge subject to a constraint on the permissible beam cross section at full energy. In Part II we describe *spiral stacking*—a method of systematic laying of successive-turn phase space ellipses along a logarithmic spiral in matched radial phase space; this results in uniformly dense packing of the beam with losses. The beam is injected above the median plane and placed along the spiral through a programmed tune shift and a programmed displacement of the equilibrium orbit.

Introduction to Part I

Here we analyze the process of multiturn injection into radial betatron phase space by obtaining simple expressions for that region of phase space A_n that can be filled on a particular turn n and will survive all subsequent apertures. The apertures considered are obtained by mapping the location of the septum for all future turns back to turn n. At the same time we obtain expressions for the phase area lost to the *septum shadow* during turn n. The object is —of course—to determine how to inject the beam in order to achieve the greatest brightness.

Geometry

The discussion is restricted to considering injection at a radial waist only. The geometry of radial phase space at the time the *n*th turn is injected is shown in Fig. 1. Here the abscissa is the displacement from the instantaneous equilibrium orbit whereas the ordinate is $\beta x'$—the product of the Twiss function with the slope relative to the equilibrium orbit. With these choices, motion from turn-to-turn is represented by a rotation about the instantaneous equilibrium orbit by an angle $\theta = 2\pi\nu$ where ν is the radial betatron frequency.

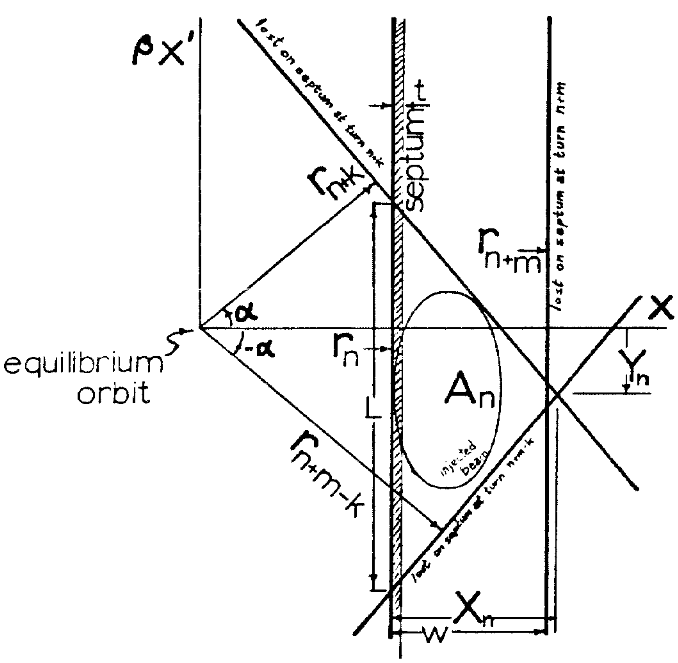

Fig. 1. Determining the Partial Acceptance for the *n*th Turn.

At turn n the inner edge of the septum corresponds to a line parallel to the $\beta x'$ axis at a distance r_n from the instantaneous equilibrium orbit. We restrict the discussion to constant betatron frequency, and specify that $r_{n+1} > r_n$ for all n.

Let the fractional part of the radial betatron frequency be the reduced fraction i/m: that is, after m turns the septum will once again be parallel to its position at turn n. Further let k be the smallest positive solution to

$$i\,k \text{ MOD } m = 1 \tag{1}$$

With the exception of the first few turns and with exception of very large m, the acceptance polygon is the intersection of the septum at turns n, n+k, n+m-k, and perhaps n+m—mapped to the phase plane at turn n. The partial acceptance A_n is an isosceles triangle except for large n where the septum at turn n+m truncates it to a trapezoid. Let $\alpha = 2\pi/m$, then the equation for the vertex of the

$$Y_n = r_{n+k}\sin\alpha - [X_n + r_n - r_{n+k}\cos\alpha]\cot\alpha = -r_{n+m-k}\sin\alpha + [X_n + r_n - r_{n+m-k}\cos\alpha]\cot\alpha \tag{2}$$

The solution is

$$X_n = \tfrac{1}{2}(r_{n+k} + r_{n+m-k})\sec\alpha - r_n \tag{3}$$

$$Y_n = \tfrac{1}{2}(r_{n+m-k} - r_{n+k})\csc\alpha \tag{4}$$

Septum Shadow

The area removed from phase space by the septum at turn n is

$$s_n = t\left[(r_{n+k} + r_{n+m-k})\csc\alpha - 2r_n\tan\alpha\right] \tag{5}$$

The remainder of the septum line is not removed from phase space at turn n. To obtain the *integrated septum shadow* we sum these contributions for all turns through turn N and rewrite the sum as

$$S_N = 2t\tan\frac{\alpha}{2}\sum_{n=1}^{N} r_n + t\csc\alpha\sum_{n=1}^{N}(r_{n+k} + r_{n+m-k} - 2r_n) \tag{6}$$

This is to be compared with the emittance generated through turn N which is $W_N = \pi r_N^2$.

A Simplification

To proceed further, it will be convenient to write

$$r_n = \Delta r\, n^{\gamma}. \tag{7}$$

We obtain the usual constant rate of shift of the instantaneous equilibrium orbit for $\gamma = 1$. Now rewrite the above equations...

$$Y_n \sim \pi\Delta r\,\gamma\left[\frac{n+(n+m)}{2}\right]^{\gamma-1}\left(\frac{2k-m}{m}\right) \tag{3'}$$

$$X_n \sim \Delta r\,\sec\alpha\left(\frac{n+n+m}{2}\right)^{\gamma} \tag{4'}$$

$$S_N \sim \frac{2t\Delta r}{\gamma+1}\,\frac{\pi}{m}\,N^{\gamma+1} \tag{6'}$$

And add *f*, the fraction of the generated emittance lost to the septum shadow:

$$f \sim \frac{t}{m\Delta r}\,\frac{2}{\gamma+1}\,N^{1-\gamma} \tag{8}$$

Observations

In general the centroid of the partial acceptance Y_n is not parallel to the equilibrium orbit but may lie at a relative angle that exceeds the divergence of the injected beam. The best injection efficiency is obtained when the ellipse is displaced by Y_n. The centroid shift is constant with turn number for $\gamma = 1$. It is smallest when turn n+k is about half way between turns n and n+m. For $\gamma = 1$, the fraction lost to the septum is constant; acceptance area is generated in proportion to n^2. The fraction lost to the can only be reduced by decreasing the ratio of the septum width to $w = m\Delta r$, the distance between homologous locations of the septum.

Now consider $\gamma = \frac{1}{2}$; acceptance is generated at a constant rate. This is desireable; however, the centroid shift Y_n and the polygon radial width now change with turn number. Thus the deflecting and focusing magnets of the injection line must be programmed to achieve optimal filling. An alternate where losses are acceptable is to choose Δr small (and N large) so as to completely saturate the partial acceptances. We see from eq. (8) that this will increase the fraction

* The work in Part I was supported by the National Bureau of Standards.
† The work in Part II was supported by the U. S. Atomic Energy Commission.

of the generated acceptance lost to the septum shadow. Thus one compromise between having excess voids due to excess partial acceptance area and having an excessive fraction of the acceptance lost to the septum shadow.

Introduction to Part II

The usual process of multiturn injection into betatron space of a synchrotron is accomplished by moving the instantaneous equilibrium orbit past the inflector at a constant rate and choosing the betatron frequencies so that most of the injected beam misses the inflector for a number of turns adequate to shift the equilibrium sufficiently away from the inflector so no possibility of further loss exists. Two consequences of this method are that a large fraction of the beam is lost during the first several injected turns whereas during latter turns the dilution of betatron phase space is large. To minimize radiation-inducing losses is necessary where high-intensity beams are used. A means of reducing phase-space dilution is desireable. The proposed system meets both of these objectives.

A specific method is here proposed that accomplishes both high-density packing and minimal losses. The method is to inject along a logarithmic spiral in matched radial betatron phase space (coordinates $x\sqrt{\beta}$ and $x'/\sqrt{\beta}$) with an ellipse of constant dimension as shown in Fig. 2. The spiral is generated so that area is swept out at a constant rate. This is achieved by varying the rate of shift of the instantaneous equilibrium orbit relative to the septum and by also varying the radial betatron frequency, which determines the rate of rotation about the equilibrium orbit. This is accomplished by pulsed deflecting magnets and pulsed quadrupole magnets.

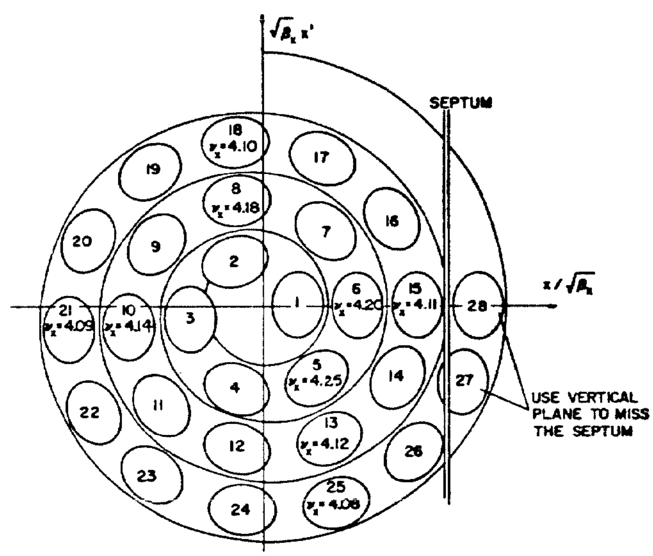

Fig. 2. Normalized Betatron Phase-space Diagram Showing Spiral Injection Scheme.

The optical components of the external injection line do not vary with time. Ellipses representing the injected beam are laid in sequence around the spiral with turn N lying between turns N-1 and N+1. As is seen from the figure, most of the previous turns would be lost on the septum were it not for an induced coherent vertical betatron oscillation that safely carries the previous turn under the corner septum[1]. The vertical betatron frequency may vary substantially as it matters only during the first turn. The vertical tune is near a half-integral value—the actual distance depends upon the height of the corner septum above the median plane. One quadrupole would suffice to provide the required . shift in radial tune were it not for resonant effects. The pulsed quadrupoles are placed at three locations such that perturb the radial betatron frequency without opening up the half-integer stop bands. These perturbations drive the vertical betatron frequency near a half integer value which is required for the missing the septum after the first turn.

Spiral Generation Equations

To meet the objective of constant rate of dilution, partial acceptance must be made available to the injected beam at a constant rate. Let r and θ be polar coordinates in the matched radial phase space shown in Fig. 2. Let the equation of the spiral be

$$r = a(\theta + \pi) \qquad [1]$$

Here a is the pitch of the spiral which should be chosen large enough to accomodate the beam plus septum width in addition to allowing some room for jitter. The rate of generation of partial acceptance is given by

$$\frac{dA}{d\theta} = 4\pi a^2\theta \qquad [2]$$

We introduce the constant k that determines the rate of generation of partial acceptance, writing

$$A = 4\pi kt + A_0. \qquad [3]$$

Here t is the time measured from some t_0, and $t_0 > 0$.

From equations [2] and [3], we find

$$\theta = \sqrt{\frac{kt}{a}} \qquad [4]$$

which then determines the radial betatron frequency to be

$$\nu_x = \frac{\tau}{a}\sqrt{\frac{k}{t}} \qquad [5]$$

with τ being the orbit period. This determines the rate of change of the pulsed quadrupole. The rate of change at the start of injection is rapid, slowing down as the injection progresses.

The separation between adjacent turns of the spiral is

$$\Delta r = 2\pi a. \qquad [6]$$

From equations [1] and [4], we obtain the rate of shifting of the instantaneous equilibrium orbit relative to the septum; this is

$$r = \sqrt{kt} + \pi a. \qquad [7]$$

This determines the rate at which the pulsed deflecting magnet is turned off.

Two constants have been introduced; k depends upon the injected beam emittance as it determines the rate partial acceptance area is generated. The width of the septum and that of the beam determine a. Tolerances on both the deflecting magnet and the pulsed quadrupole are obviously eased if these constants are increased somewhat.

Some Problems

This method requires that the radial tune approach an integer resonance. It will be necessary to be very careful to avoid opening up the stop band and destroying the stability of the equilibrium orbit. On the other hand, the tune does not stay near the resonance very long.

Similar comments apply to the vertical tune which must lie near a half integral resonance. To avoid large swings in vertical tune while shifting the radial tune according to equation [5], the pulsed quadrupole magnets must be placed where β_x is large, but β_y is small.

It is to be expected that in the process of perturbing the radial tune , the radial betatron envelope will also be perturbed. For example, using a single quadrupole to force a shift $\Delta\nu$. the radial amplitude function is perturbed according to

$$\frac{\beta}{\beta_0} \sim 1 - \pi\,\Delta\nu\,\csc \pi\nu_0 \qquad [8]$$

For the scheme shown in Fig. 2, this corresponds to about a 15% change in the betatron amplitude; such a change must obviously be reflected in the drawing of the spiral.

The fast rate of change required from the pulsed quadrupole at the beginning of the injection may pose extreme electrical power difficulties.[2] Perhaps pulsed electrostatic quadrupoles could be used to provide the tune perturbation.

No mention has been made relative to off-energy particles. They should follow their own spirals about their instantaneous equilibrium orbit provided the chromaticity is such that the spiral equations (above) are satisfied.

References

[1] R. T. Avery & P. F. Meads, Lawrence Berkeley Lab. Report LRL-905 (1972)

[2] R. T. Avery, private communication.