NEGATIVE MASS INSTABILITY IN HOLLOW CYLINDERS

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Introduction

In analyses of the negative mass instability for electron rings[1,2] it is assumed that the azimuthal electric field, E_θ, and the perturbed charge density are proportional. The proportionality factor depends on the detailed structure of the ring and tank and with proper normalization is referred to as the geometry factor. Calculations of this geometry factor are reported here for an electron ring in a perfectly conducting, infinite cylinder in a manner which takes into account the shape of the ring and the image effects due to the walls. This is done by employing a normal mode expansion for the electromagnetic fields in the cylinder in the form of a dyadic Green's function which satisfies the appropriate boundary conditions on the walls of the cylinder. The calculations are carried out explicitly for a ring of major radius, R_o, centered on the cylinder's axis and with rectangular cross-section and uniform charge density. The value of E_θ at R_o is used to determine the geometry factor, and this geometry factor is then used to investigate the negative mass instability following the formulation of Landau and Neil.[2] The analysis yields both real and complex values for the geometry factor.

The Azimuthal Electric Field

In this analysis a solution is sought for the electric field due to perturbations on a relativistic electron ring apropos of the E. R. A. configuration. The equilibrium fields will not be considered. The perturbations are assumed to be harmonic in time, at frequency ω, in which case the electric field is a solution to the vector wave equation:

$$\bar{\nabla} \times \bar{\nabla} \times \bar{E} - k^2 \bar{E} = j\omega\mu_o \bar{J}(\bar{r}) \tag{1}$$

where $k = \omega/c$, $\bar{J}$ is the current density due to the perturbations, and μ_o is the permeability of vacuum. Equation 1 can be integrated formally in a manner analagous to that employed for the scalar wave equation[3] with the aid of a dyadic Green's function, $\bar{\bar{G}}(\bar{r}/\bar{r}')$, which is a solution of:

$$\bar{\nabla} \times \bar{\nabla} \times \bar{\bar{G}}(\bar{r}/\bar{r}') - k^2 \bar{\bar{G}}(\bar{r}/\bar{r}') = -\delta(\bar{r}-\bar{r}')\bar{\bar{I}} \tag{2}$$

where $\bar{\bar{I}}$ is the unit dyadic. Following such a procedure[4], a solution to (1) can be obtained which satisfies the boundary condition $\bar{E} \times \hat{n} = 0$ on a perfectly conducting boundary surface, S, with outward normal, $\hat{n}$, in the form:

$$\bar{E}(\bar{r}) = j\omega\mu_o \iiint \bar{\bar{G}}(\bar{r}/\bar{r}') \cdot \bar{J}(\bar{r}')\, d\bar{r}' \tag{3}$$

where $\bar{\bar{G}}(\bar{r}/\bar{r}')$ is a solution of (2) which satisfies the boundary condition $\hat{n} \times \bar{\bar{G}}(\bar{r}/\bar{r}') = 0$ on S. The required dyadic Green's function is known for hollow, perfectly conducting cylinders and will be used below to calculate the azimuthal electric field due to perturbations on an electron ring enclosed in the cylinder.

In the specific case apropos of the negative mass instability, the current density is due to the perturbed electrons and is assumed to have the form:

$$\bar{J}(r,\theta,z,t) = \hat{\theta}\, J(r,z)\, e^{j(N\theta-\omega t)} \tag{4}$$

$$= \hat{\theta}\, (\omega r/N)\, \rho(r,z) e^{j(N\theta-\omega t)} \tag{5}$$

(5) being a consequence of conservation of charge applied to the charge density, $\rho(r,z,\theta,t)$. It follows that the θ-component of electric field is:

$$E_\theta(r,\theta,z,t) = j(\omega^2\mu_o/N) \int \left\{ \hat{\theta}\cdot\bar{\bar{G}}(\bar{r}/\bar{r}')\cdot\hat{\theta}\; r\rho(r,z) e^{j(N\theta-\omega t)} \right\} d\bar{r}' \tag{6}$$

The dyadic Green's function for the interior of a perfectly conducting cylinder which satisfies the necessary boundary condition is known,[4] and its azimuthal component, $\hat{\theta}\cdot\bar{\bar{G}}(\bar{r}/\bar{r}')\cdot\hat{\theta}$, takes the form:

$$\theta \cdot \bar{\bar{G}}(\bar{r}/\bar{r}') \cdot \theta = j/4\pi \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (2-\delta_o) \left\{ \frac{J_n'(\mu r) J_n'(\mu r')}{\mu^2 I_\mu K_\mu} \cos n(\theta-\theta') e^{\pm jK_\mu(z-z')} + (n^2 K_\lambda / k^2 I_\lambda) \frac{J_n(\lambda r)\, J_n(\lambda r')}{r\, r'} \cos n(\theta-\theta')\, e^{\pm jK_\lambda(z-z')} \right\} \tag{7}$$

where $(\pm)$ applies to $z \gtrless z'$, respectively, and $(')$ denotes differentiation with respect to the argument. the parameters μ, λ are the zeros of the Bessel functions $J_n(x)$ and $J_n'(x)$ respectively divided by the cylinder's radius, a, and for each n there are m zero's.* The other parameters in (5) are:

$$I_\mu = \frac{1}{2}\left[a^2 - (n/\mu)^2\right] J_n^2(a)$$

$$I_\lambda = \frac{1}{2}(a/\lambda)^2 \left[J_n'(\lambda a)\right]^2$$

$K_\mu = \sqrt{k^2 - \mu^2}$

$K_\lambda = \sqrt{k^2 - \lambda^2}$

$$\delta_o = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

The θ-integration, in (7) is readily performed to yield the following expression for E_θ at the ring's center ($r = R_o$ and $z = 0$) when $N > 0$:

$$E_\theta = -e^{j(N\theta - \omega t)} \sum_{m=1}^{\infty} \iint \rho(r', z') \Big[Q_m r' J_n'(\mu r') e^{\pm jK_\mu z'} + P_m J_n(\lambda r') e^{\pm jK_\lambda z'} \Big] r' dz' dr' \quad (8)$$

where:

$$Q_m = (\omega^2 \mu_o / 4N\mu^2 I_\mu K_\mu) J_N' (\mu R_o)$$

$$P_m = (\omega^2 \mu_o N K_\lambda / 4k^2 \lambda^2 I_\lambda R_o) J_N(\lambda R_o)$$

The remaining integration requires specification of the cross-section and the electron density, $\rho(r,z)$, of the ring. Assuming that the ring has rectangular cross-section with constant density, ρ_o, so that $\rho(r,z) = \rho_o$ when $R_1 \le r \le R_2$ and $-\xi < z < \xi$, and is zero otherwise, one obtains:

$$E_\theta = -(j/N\varepsilon_o)\rho_o e^{j(N\theta - \omega t)} \sum^{\infty} \Big\{ \frac{Q_{Nm}}{\mu^3} \Big[G_N^{(1)}(\mu) - 2G_{N+1}(\mu) - 2N F_{N+1}(\mu) \Big] + \frac{P_{Nm}}{\lambda^2} \Big[G_{N+1}(\lambda) + N F_{N+1}(\lambda) \Big] \Big\} \quad (9)$$

where:

$$Q_{Nm} = \Big[k^2 / 2\mu I_\mu K_\mu^2 \Big] \Big[1 - e^{jK_\mu \xi} \Big] J_N'(\mu R_o)$$

$$P_{Nm} = \Big[N^2 / 2\lambda^2 I_\lambda R_o \Big] \Big[1 - e^{jK_\lambda \xi} \Big] J_N(\lambda R_o)$$

$$G_N(\alpha) = \alpha \Big[R_2 J_N(\alpha R_2) - R_1 J_N(\alpha R_1) \Big]$$

$$G_N^{(1)}(\alpha) = \alpha^2 \Big[R_2^2 J_N(\alpha R_2) - R_1^2 J_N(\alpha R_1) \Big]$$

and where $F_N(\alpha)$ is given by

$$J_o(\alpha R_2) - J_o(\alpha R_1) - 2 \sum_{s=0}^{(N-1)/2} \Big[J_{2s}(\alpha R_2) - J_{2s}(\alpha R_2) \Big]$$

when N is odd, and by

$$\int_{\alpha R_1}^{\alpha R_2} J_o(\alpha x) dx - 2 \sum_{s=0}^{(N-2)/2} \Big[J_{2s+1}(\alpha R_2) - J_{2s+1}(\alpha R_1) \Big]$$

when N is even.

Negative Mass Instability

Following the procedure of Landau and Neil,[2] the dispersion equation for the negative mass instability is:

$$1 = (q^2 R_o G/N) \int \frac{\partial F_o / \partial p}{(\omega/N) + Kp - \omega_o} dp \quad (10)$$

where $K = \alpha / \gamma M R_o^2$, $\alpha = \Big[\frac{1}{1-n} - \frac{1}{\gamma^2} \Big]$, and n is the field index, M is the rest mass of electrons and ω_o is the equilibrium rotational frequency of the electrons. The geometry factor, G, is defined by the equation, $E_\theta = jG\rho$; hence, G is equal to $-1/N\varepsilon_o$ times the summation in (9). Assuming a square equilibrium distribution function, F_o, of width, Δ, (10) can be integrated to yield the following expression for the allowed frequencies, $\omega^{(2)}$:

$$\omega/N = \omega_o \pm \frac{\alpha}{2R_o} \Big[(\Delta/\gamma M R_o)^2 - (\omega_p R_o / \pi \alpha N^2)^2 \tilde{G} \Big]^{\frac{1}{2}} \quad (11)$$

where $\tilde{G} = -N\varepsilon_o G$. Thus $\tilde{G}$ is equal to the summation in (9). Equation 9 is coupled to (11) through the frequency dependent factor, k, making a general solution for $\tilde{G}$ difficult; however, in many cases the radical in (11) is quite small compared to ω_o so that the assumption $\omega = N\omega_o$ can reasonably be made and justified a posteriori. This assumption has been made here and the summation in (8) subsequently evaluated numerically to yield $\tilde{G}$.

Some representative values of $\tilde{G}$ obtained as outlined above are shown in Tables I and II for the case: a = .15m, ξ = .007m and $R_2 - R_1$ = .004m. The data in Table I are for N = 1 and those in Table II for N = 3, in each case shown as a function of mean ring radius, R_o.

Table I
The Geometry Factor, $\tilde{G} \times 10^3$
N = 1

Re $\tilde{G}$	Im $\tilde{G}$	R_o
.2720	-.4585	.01
-.0287	-.0694	.03
.0721	-.0613	.05
.0561	-.0968	.07
-.0483	.0000	.09
-.0082	.0000	.11
-.0012	.0000	.13

Table II
The Geometry Factor, $\tilde{G} \times 10^3$
N = 3

Re $\tilde{G}$	Im $\tilde{G}$	R_o
1.0710	-1.0200	.01
.4806	- .6293	.03
.2816	- .2837	.05
.0840	- .1243	.07
.2463	- .2325	.09
-.3955	.0000	.11
-.0152	.0000	.13

Additional values for a ring of the same cross-section as above but with a tank radius of a = .25m are shown in Table III.

Table III
The Geometry Factor, $G \times 10^3$
N= 1

Re G	Im G	R_o
.2491	-.3173	.01
.0748	-.0788	.04
.0413	-.0312	.07
.0400	-.0488	.10
.0276	-.0970	.13
-.0157	.0000	.16
-.0062	.0000	.19
-.0009	.0000	.22

The values shown in Tables I-III are typical in that the geometry factor is complex for small radii, and real and negative for larger values of mean ring radius. The change from complex to real, negative values of geometry factor occurs at a critical radius, $R_o = Naq_{Nl}^{-1}$, as can be seen from the definitions of Q_{Nm} and P_{Nm} as used in (9). Trends not illustrated by the examples in Tables I-III are a strong dependence of $\tilde{G}$ on cross-sectional area but slight dependence on cross-sectional shape. Thus, $\tilde{G}$ increases roughly proportional to increases in cross-sectional area, but changes only slightly, with the dimensions of the rectangle: Re$\tilde{G}$ appears to increase and Im$\tilde{G}$ to decrease as the cross-section is elongated in the z-direction from an initial square, but this change is small.

When $\tilde{G}$ is complex (9) will always yield complex ω regardless of momentum spread (presumed real); and consequently, the negative mass instability will have no absolutely stable regime. This is in contrast to the case in which $\tilde{G}$ is real and positive[1,2,5] for which there is a critical Δ above which ω is real. The effect of complex values of $\tilde{G}$ on the negative mass instability is illustrated in Table IV in which the growth rate of the instability as predicted by (11) is shown as a function of electron density and momentum spread. The growth rate is the time required for the amplitude of the perturbation to increase by a factor, e , and is expressed in μ-sec. in the table. The momentum spread is given in % and the electron density in number/length. The data in Table IV were prepared from a geometry factor corresponding to the case: a = .15m, R_o = .05m, [illegible] = .007, $R_1 - R_2$ = .004, N = 1, n = 0 and γ = 11. The growth rate predicted for this ring when it contains 3.14×10^{13} electrons with 1% momentum spread is about 33 μ-sec.

Although no absolutely stable regime exists for complex $\tilde{G}$, the perturbations are always stable (i.e. regardless of momentum spread) when the real part of $\tilde{G}$ is negative. The data in Table I - III indicate that for rings with mean radius greater than the critical radius, that the ring is stable against the negative mass type of instability. It is interesting to note that the critical radius corresponds to the "cut-off" conditions in cylindrical wave guides. Since, at

Table IV
Growth Rate in μ-sec. as a Function of Momentum Spread and Electron Density

Momentum Spread (percent)	Electron Density (number/length)			
	10^{13}	5×10^{13}	10^{14}	5×10^{14}
0.5	165	33.3	16.5	3.33
1.0	333	66.5	33.3	6.65
5.0	1650	333.0	165.0	33.30
10.0	3330	665.0	333.0	66.50
15.0	5000	1000.0	500.0	100.00

constant energy, the rotational frequency of the ring is inversely proportional to radius, it follows that the frequency, $\omega \cong N\omega_o$, associated with a given negative mass mode, decreases with increased ring radius. When $R_o > Naq_{Nl}^{-1}$, all of the "modes" associated with hollow cylinders are evanescent at the corresponding $N\omega_o$; when $R_o < Naq_{Nl}^{-1}$, at least one mode is propagating; and the condition $R_o = Naq_{Nl}^{-1}$ corresponds to the cut-off condition of the TE_{Nl} mode in the cylinder, the mode with the lowest cut-off frequency.

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References

1. C. E. Nielsen, A. M. Sessler, K. R. Symon, "Longitudinal Instabilities in Intense Relativistic Beams," CERN Proc., pp. 235-252, Geneva 1959.

2. R. W. Landau and V. K. Neil, "Negative Mass Instability," Phys. Fluids, 9(12), pp. 2412-2427, 1966.

3. J. D. Jackson, Classical Electrodynamics, John Wiley and Sons, 1969.

4. Chen-To Tai, Dyadic Green's Functions in Electromagnetic Theory, International Textbook Co., 1971.

5. C. E. Nielsen and A. M. Sessler, "Longitudinal Space Charge Effects in Particle Accelerators," Rev. Sci. Inst., 30 (2), pp. 80-89, 1959.

Footnote

[1] If the zero's of $J_n(x)$ and $J_n'(x)$ are p_{nm} and q_{nm}, respectively, then $\lambda = p_{nm}/a$ and $\mu = q_{nm}/a$.