ENVELOPE INSTABILITIES IN RELATIVISTIC ELECTRON RINGS

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Introduction

The possibility of resonant coupling of the transverse degrees of freedom of concentric electron and ion rings has recently been demonstrated for the case of stationary ions and relativistic electrons apropos of the E. R. A. configuration.[1] In contrast to the "dipole" or "kink" instability, this transverse instability results from perturbation in the ring's cross-section with fixed center of mass and exhibits unstable bands as a function of electron and ion density.[1,2]

The existing analysis assumes beams of circular cross-section and neglects one of the transverse degrees of freedom in order to obtain a tractable dispersion equation. Extention of this analysis is reported here to include the case of elliptic cross-section, and numerical solutions are obtained of the resultant dispersion equation including all of the degrees of freedom. The extended analysis also yields band-like regions of instability and is in agreement with the previous work for the special case of circular cross-section; however, in the case of elliptic cross-section, the bands shift their position and shape relative to those at circular cross-section and split into pairs of bands where only one was reported previously. The motion of the bands as a function of cross-section is such as to suggest the possibility that this instability may be self-stabilizing.

Linearized Envelope Equations

Following the development of Koshkarev and Zenkevich[1] the single particle equations of motion for each degree of freedom are written, transformed to the corresponding envelope equations and linearized. This is done under the assumption of concentric beams of stationary ions and moving electrons, each with constant density and zero momentum spread and in local equilibrium. Hence, the dominant forces on a particle subject to a small perturbation are an electrostatic restoring force due to particles of the other species and the external magnetic focussing forces.[3] The force balance for each transverse degree of freedom yields:

$$\ddot{y}_e + \omega_o^2 n_y^2 y_e = -w^2 y_e [g_{y_i}(g_{y_i}+g_{z_i})]^{-1} \quad (1a)$$

$$\ddot{y}_i = -\xi w^2 y_i [g_{y_e}(g_{y_e}+g_{z_e})]^{-1} \quad (1b)$$

$$\ddot{z}_e + \omega_o^2 n_z^2 z_e = -w^2 z_e [g_{z_i}(g_{z_i}+g_{y_i})]^{-1} \quad (1c)$$

$$\ddot{z}_i = -\xi w^2 z_i [g_{y_e}(g_{y_e}+g_{z_e})]^{-1} \quad (1d)$$

where:

$w^2 = q^2 N_i/\gamma M_e \epsilon_o$

$\xi = \gamma M_e/fM_i$

$f = N_i/N_e$

N_i = number of ions/M^3

N_e = number of electrons/M^3

ϵ_o = permitivitty of vacuum

and where y and z are coordinates along the radial and axial directions respectively, the $g_{y,z}$ are the semi-axis of the cross-sectional ellipses, ω_o is the equilibrium rotational frequency of the electrons, and $n_{y,z}$ are the field indices in the radial and axial directions, respectively.

In the case of a microcanonical distribution in phase space, (1) may be transformed to the corresponding envelope equations.[1,4] Assuming small perturbations, $\eta_{e,i}$ and $\xi_{e,i}$, from the equilibrium cross-section in the y and z directions respectively, and assuming that the equilibrium electron and ion ellipses are identical, each with semi-major axis, F_o, and semi-minor axis, G_o, the envelope equations, to first order in the perturbations are:

$$\ddot{\eta}_e + 4(\omega_o^2 n_y^2 + \tilde{Q}^2)\eta_e - a\beta\tilde{Q}^2\eta_i - \beta\tilde{Q}^2\xi_i = 0 \quad (2c)$$

$$\ddot{\xi}_e + 4(\omega_o^2 n_z^2 + \tilde{Q}^2)\xi_e - b\beta\tilde{Q}^2\xi_i - \beta\tilde{Q}^2\eta_i = 0 \quad (2b)$$

$$\ddot{\eta}_i + 4\xi\tilde{Q}^2\eta_i - a\beta\xi\tilde{Q}^2\eta_e - \beta\xi\tilde{Q}^2\xi_e = 0 \quad (2c)$$

$$\ddot{\xi}_i + 4\alpha\xi\tilde{Q}^2\xi_i - b\beta\xi\tilde{Q}^2\xi_e - \beta\xi\tilde{Q}^2\eta_e = 0 \quad (2d)$$

where:

$\tilde{Q}^2 = w^2/\pi\alpha(\alpha+1)F_o^2$

$\alpha = G_o/F_o$

$a = 2 + 1/\alpha$

$b = 2 + \alpha$

and where the densities $N_{e,i}$ are now expressed as number per unit length.

Assuming that the perturbations have the form:

$$\eta_{e,i} = n_{e,i}\, e^{j(N\epsilon-\omega t)} \quad (3a)$$

$$\xi_{e,i} = x_{e,i}\, e^{j(Ne-\omega t)} \tag{3b}$$

one obtains the following algebraic equations for the amplitude, $n_{e,i}$ and $x_{e,i}$ of the perturbation:

$$\Delta_e n_e + 0 - an_i - x_i = 0 \tag{4a}$$

$$0 + \not\Delta_e x_e - n_i - bx_i = 0 \tag{4b}$$

$$-a\,{}_e - x_e + \Delta_i n_i + 0 = 0 \tag{4c}$$

$$-n_e - bx_e + 0 + \not\Delta_i x_i = 0 \tag{4d}$$

where:

$$\Delta_e = [-(v-N)^2 + 4(n_y^2+Q^2)]/\beta Q^2$$

$$\not\Delta_e = [-(v-N)^2 + 4(n_y^2+Q^2)]/\beta Q^2$$

$$\Delta_i = [-v^2 + 4\xi Q^2]/\xi\beta Q^2$$

$$\not\Delta_i = [-v^2 + 4\alpha\xi Q^2]/\xi\beta Q^2$$

and $Q = \tilde{Q}/\omega_o$ and $v = \omega/\omega_o$.

Equations 4 are similar to those obtained by Koshkarev and Zenkevich[1] except that they include the case of elliptical cross-section. Equations 4 reduce to their results when $\alpha = 1$.

The Dispersion Equation

Equations 4 admit non-trivial solutions only when the determinant of the coefficients is zero. This requirement leads to the following dispersion equation for the allowed frequencies, v :

$$\sum_{j=0}^{8} C_j v^j = 0 \tag{5}$$

where:

$$C_o = \alpha S^2 TV + RS\left[(1+\alpha a^2)V + (\alpha+b^2)T\right] + R^2(ab-1)^2$$

$$C_1 = -2NS\left\{\alpha S(T+V) + R\left[\alpha(1+a^2) + 1 + b^2\right]\right\}$$

$$C_2 = S\left[4\alpha N^2 S - T(V-\alpha S) - \alpha V(T-S)\right] - R\left[(1+b^2)(T-S) + (1+a^2)(V-\alpha S)\right]$$

$$C_3 = 2N\left\{S[(1+\alpha)(T+V) - 2\alpha S] + R(2+a^2+b^2)\right\}$$

$$C_4 = -S\left[T+V+4N^2(\alpha+1)\right] + (T-S)(V-\alpha S) - R(2+a^2+b^2)$$

$$C_5 = 2N\left[2(\alpha+1)S - T - V\right]$$

$$C_6 = 4N^2 + T + V - (\alpha+1)S$$

$$C_7 = -4N$$

$$C_8 = 1$$

and where:

$$T = N^2 - 4n_y^2 - 4Q^2$$

$$V = N^2 - 4n_z^2 - 4\alpha Q^2$$

$$S = 4\xi Q^2$$

$$R = \xi\beta^2 Q^4$$

In general, this dispersion equation is a difficult eight order polynomial; however, there are a few special cases in which it can be reduced to a quartic of the following form:*

$$[(v-k)^2 - Q^2 - n_z^2][v^2 - \xi Q^2] - p\xi Q^4 = 0 \tag{6}$$

One such case is that of complete symmetry ($\alpha = 1$ and $n_z = n_y$) for which (5) factors into a pair of quartics in the form of (6) with $k = N/2$ and $p = (5 \pm 3)/32$, and as pointed out by Koshkarev and Zenkevich these cases correspond to symmetric and anti-symmetric modes of oscillation and admit bands in N_e , N_i space in which the perturbations are unstable.[1]

Equation 6 also applies to the approximate solution proffered by Koshkarev and Zenkevich in which they assumed that perturbations in the radial direction of the electron beam will be neglible, set $n_e = 0$ and neglected the equivalent of (4a). The result is (6) with $k = N/2$ and $p = 5/32$, which also admits bands of unstable perturbation in density space.[1,2] Unfortunately, this approximation is somewhat unsatisfying because it does not reduce to either of the quartics applicable to the completely symmetric case when $n_z = n_y$, but rather to some compromise between them (i.e. $p = 5/32$ instead of either $p = (5 \pm 3)/32$). Nor does the approximate solution offer a means to account for variation in radial field index, n_y , or changes in the cross-section. Furthermore, since it is described by a polynomial of only fourth order, it may neglect possible unstable modes.

Results

The general dispersion equation (5) has been solved numerically for various combinations of the field indices and cross-section as a function of electron and ion density. Examples of these results are shown in Figures 1 and 2 for $n_z = 0$, $n_y = 1$, $v = 11$, $N = 1$ and a mean ring radius of .10 meters.

In the case of circular cross section (Figure 1, $\alpha = 1$) the solution to (5) appears to be in general agreement with the approximate disperion equation of Koshkarev and Zenkevich[1,2], indicating that the approximation is a reasonable one in this case.

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*Equation 6 also applies to the "dipole" instability when $p = 1$ and $k = N$[5]. This is the degenerate case in which there is no band structure to the unstable region. Rather, for each N_e there is a maximum N_i which a stable beam may have.

However, the evidence suggests that the single band predicted by the quartic is not really one band but two over lapping bands. This follows from analysis of the completely symmetric case in which the dispersion equation factors into two quartics. But the two bands do not appear to be manifested unless $\alpha \neq 1$.

In the case of elliptic cross-section, the band structure predicted by (5) differs considerably from that predicted by the approximate solution. This is evident in Figure 1 where the case $\alpha = 3$, and in Figure 2 where the case $\alpha = 1/10$ is plotted. Notice that in each case there are two parts to each band and that the bands are shifted down for $\alpha > 1$ and up for $\alpha < 1$. In addition, the band structure is broader for $\alpha < 1$ and narrower for $\alpha > 1$ than in the case of circular cross-section.

The shifting of the bands as the beam cross-section changes suggests the possibility that this instability may be self-stabilizing. For example, consider a beam of initial circular cross-section for which the beam parameters place it initially in an unstable band with $n_z < n_y$. It seems reasonable to assume that the cross-section will grow more rapidly in the z-direction since the focussing forces are weakest in this direction. If this is so, α will decrease with time and the unstable region will move in the direction of larger Q and ξQ as indicated in Figure 2. The parameters Q and ξQ are proportional to $[G_o(G_o+F_o)]^{-1/2}$ at constant density per unit length, so that as the beam's cross-section grows, Q and ξQ will decrease. Thus, in density space the beam's position and the unstable band may move in different directions, and it is possible that as a result of this motion the beam will move into a stable region.

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References

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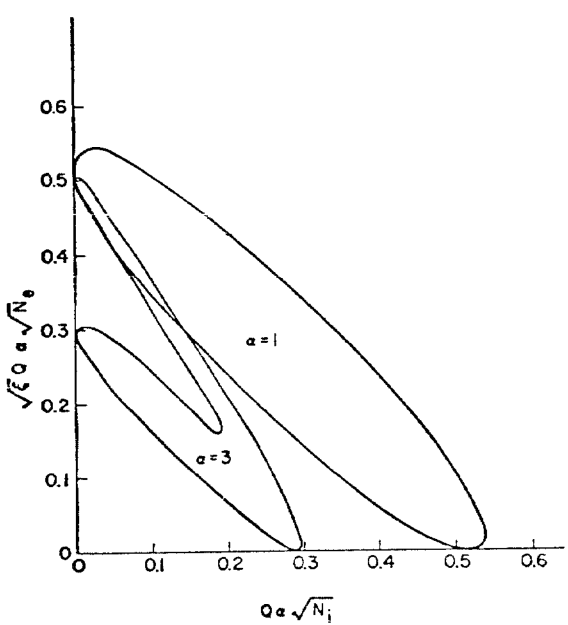

Figure 1. Unstable regions for the cases $\alpha = 1$ and $\alpha = 3$.

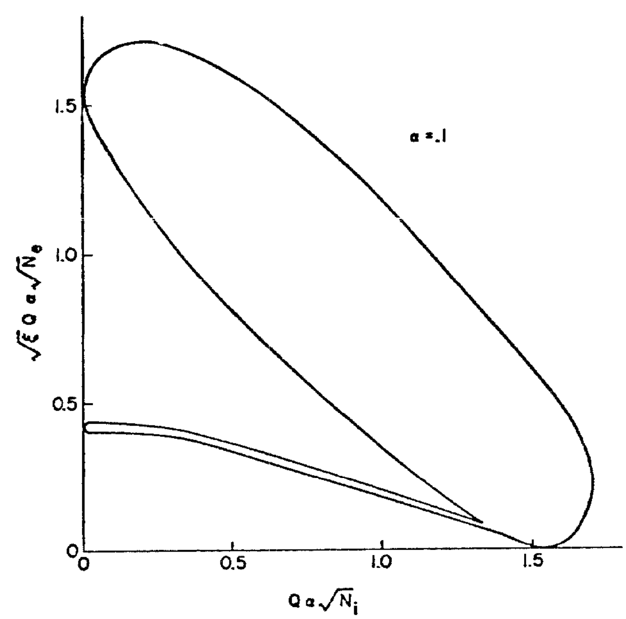

Figure 2. Unstable regions for the case $\alpha = .1$.