SELF FIELDS AND PARTICLE OSCILLATION FREQUENCIES IN A ROTATING ELECTRON SHEET BEAM LOADED WITH POSITIVE IONS *

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Summary

An approximate analytical treatment is presented for the electric and magnetic self fields of a long rotating sheet beam between coaxial conductors. By suitable choice of the conductor radii and/or by apply ing a voltage between the conductors it is possible to shift the potential minimum to the major radius of the beam; the condition for this is derived. Expressions for the equilibrium fields and gradients and for the particle oscillation frequencies in various geometries are presented.

Introduction

The effects of the self fields of a toroidal electron beam on the equilibrium orbit and the linear particle oscillation frequencies have been calculated by several authors in connection with electron ring accelerator work. [1,2,3] In these studies it was assumed that (a) the ring is in free space, (b) the ratio of the average minor radius to major radius R is small compared to unity, (c) the particle and current density is constant over the cross section, and (d) the density of stationary positive ions trapped in the ring is proportional to that of the electrons. Laslett also estimated the influence of a boundary at close distance from the beam [2]. Under these conditions the expressions for the square of the radial and axial oscillation frequencies, ν_r^2 and ν_z^2, can be represented as a sum of the focusing effect due to the applied magnetic field, a toroidal or "bias" term, a "straight beam" term, and the image-field effects.

As was pointed out by this author [3], the assumption of proportionality between ion and electron density distribution is not quite justified. The radial force due to the "bias" field separates the centers of mass of the electron and ion distribution and one gets an undesirable polarization which reduces the holding power and may lead to the dipole instabilities that were treated theoretically by Koshkarev and Zenkevich [4]. A possible solution to this problem is to employ a radial electric field which shifts the negative potential minimum to the major radius. In this case, the center of mass of the two subrings coincide. Such an electric field can be provided by placing a cylindrical conducting rod inside and a coaxial conducting boundary outside the beam. The condition that electric field $E_r = 0$ at $r = R$ can then be fulfilled either by a suitable choice of the radii of the two conductors or by applying a potential difference between them, or a combination of both. A theoretical treatment of this problem involves elliptical integrals with the appropriate boundary conditions and numerical analysis by a suitable computer program. However, the mathematical complexity of this problem can be substantially reduced if one assumes that the axial length $\Delta z = L$ of the ring beam is large compared to the major radius R. In this case, the field distribution in the midplane ($z = 0$) can be calculated analytically in a straightforward way by a two-dimensional approximation. The results of such an analysis are presented in this paper. They are directly applicable to Astron-type E-layers, hollow rotating sheet beams [5], and the beam in the University of Maryland ERA experiment prior to full axial compression. In addition, they can serve as a useful guide for accurate computer studies of rings with small cross section.

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Geometry and Electric Field

The geometry of the rotating beam between coaxial cylindrical conductors is illustrated in Fig. 1. A potential difference $V_b - V_a$ is applied between the inner and outer conductor, and a uniform external magnetic field aids in keeping the electrons on a circular path. The actual electromagnetic field distribution is then a superposition of the external fields and the self fields of the beam (including the induced image charges and currents in the conductors). It is assumed that the magnetic self field has not penetrated into the conductors, i.e., we consider a quasi-stationary situation on a time scale which is short compared to the decay time for the image currents. The beam consists of N_e electrons rotating at uniform azimuthal velocity $v_o = \beta c$, and $N_i = fN_e$ stationary ions. Charge and current density are assumed constant over the cross section of the beam. The model is thus not self-consistent, but it will provide a good approximation for relativistic beams with small energy spread and low ν/γ value (as defined in the last section). The axial extent of the beam, L, is assumed to be large compared to the major radius $R = (r_1 + r_2)/2$ to permit a two-dimensional analysis for the field distribution in the midplane (z=0).

Charge density is defined by

$$\rho = -en = -en_e(1-f) = -\frac{eN_e(1-f)}{L\pi(r_2^2 - r_1^2)} \quad \text{for } r_1 \le r \le r_2 \tag{1}$$

$\rho = 0$ for $a < r < r_1$ and $r_2 < r < b$.

Current density is then given by

$$J_\theta = en_e \beta c = \frac{eN_e \beta c}{L\pi(r_2^2 - r_1^2)} \quad \text{for } r_1 \le r \le r_2 \tag{2}$$

$J_\theta = 0$ for $a < r < r_1$ and $r_2 < r < b$

Total current is

$$I = J_\theta (r_2 - r_1) L = \frac{eN_e \beta c}{\pi(r_1 + r_2)} \tag{3}$$

To find the electric potential distribution $V(r)$ in the midplane we solve Poisson's Equation

$$\nabla^2 V = \frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = -\frac{\rho}{\epsilon_o} \tag{4}$$

for each region subject to the conditions $V = V_a$ at $r = a$, $V = V_b$ at $r = b$, $V(r)$ and $V'(r)$ continuous at $r = r_1$ and $r = r_2$. The results are as follows:

Region I ($a < r < r_1$):

$$\frac{dV}{dr} = V' = \frac{C_1}{r} \tag{5}$$

$$V = V_a + C_1 \ln \frac{r}{a} \tag{6}$$

Region II ($r_1 \le r \le r_2$):

$$V'' = -\frac{C_1}{r^2} + \frac{en}{2\epsilon_o}\left[1 + (r_1/r_2)^2\right] \tag{7}$$

$$V' = \frac{C_1}{r} + \frac{en}{2\epsilon_o}(r_2^2 - r_1^2)\frac{1}{r} \tag{8}$$

$$V = V_a + \frac{en}{2\epsilon_o}\left[\frac{1}{2}(r^2 - r_1^2) - r_1^2 \ln\frac{r}{r_1} + C_1 \ln\frac{r}{a}\right] \tag{9}$$

Region III ($r_2 < r < b$):

$$V' = \frac{C_1}{r} + \frac{en}{2\epsilon_o}(r_2^2 - r_1^2)\frac{1}{r} \tag{10}$$

$$V = V_b - C_1 + \frac{en}{2\epsilon_o}(r_2^2 - r_1^2)\frac{1}{r}\ln\frac{b}{r} \tag{11}$$

The constant C_1 is given by

$$C_1 = \frac{1}{\ln(b/a)}\left\{V_b - V_a - \frac{en}{2\epsilon_o}\left[(r_2^2 - r_1^2)\frac{1}{2} + r_2^2 \ln\frac{b}{r_2} - r_1^2 \ln\frac{b}{r_1}\right]\right\}. \tag{12}$$

Of particular interest is the case where the potential minimum occurs within the region of the beam ($r_1 \le r \le r_2$). The radius r_o for which V is a minimum or $V' = 0$, is obtained from Eq. (8):

$$r_o = \left[r_1^2 - \frac{2\epsilon_o}{en} C_1\right]^{1/2} \tag{13}$$

To avoid the polarization effect discussed in the introduction, one would like to have the minimum occur at the center of the beam, i.e., $r_o = R$, or

$$C_1 = \frac{en}{2\epsilon_o}(R^2 - r_1^2) \tag{14}$$

For given n, r_1, r_2, Eq. (14) then determines the potential difference $V_b - V_a$ and the conductor radii a and b to satisfy this condition.

If the radial width $\Delta r = r_2 - r_1$ of the beam is small compared to the major radius R, (thin layer), one can expand the functions (7) to (9) and (12) to (14) about the radius R and neglect all nonlinear terms in x/R and δ/R, where we define:

$$r = R + x = R\left(1 + \frac{x}{R}\right) \tag{15}$$

$$r_2 - r_1 = 2\delta, \quad r_2 = R + \delta \tag{16}$$

The linear approximation ($\frac{x}{R} \ll 1$, $\frac{\delta}{R} \ll 1$) then leads to the following expressions for the constant C_1, the potential $V(R)$ and its first and second derivatives at $r = R$ ($x = 0$):

$$C_1 = \frac{1}{\ln(b/a)}\left[V_b - V_a - \frac{2\,en}{\epsilon_o} R\,\delta \ln(b/R)\right] \tag{17}$$

$$V(R) = V_a + C_1 \ln\frac{R}{a} \tag{18}$$

$$V'(R) = \frac{1}{\ln(b/a)}\,\frac{V_b - V_a}{R} + \frac{en\,\delta}{\epsilon_o}\left[1 - 2\frac{\ln(b/R)}{\ln(b/a)}\right] \tag{19}$$

$$V''(R) = -\frac{V_b - V_a}{R^2 \ln(b/a)} + \frac{en}{\epsilon_o}\left[1 + 2\frac{\ln(b/R)}{\ln(b/a)}\frac{\delta}{R} - \frac{\delta}{R}\right] \tag{20}$$

The desired condition that V is a minimum at $r = R$, or $V'(R) = 0$ leads to

$$C_1 = \frac{-enR}{\epsilon_o}, \tag{21}$$

from Eq. (14) for $\delta/R \ll 1$,

$$V_b - V_a = \frac{enR\delta}{\epsilon_o}\left[2 \ln\frac{b}{R} - \ln\frac{b}{a}\right], \tag{22}$$

from Eq. (19) (with $V' = 0$).

Furthermore

$$V''(R) = \frac{en}{\epsilon_o} \tag{23}$$

and

$$V(R) = V_a - \frac{enR\delta}{\epsilon_o}\ln\frac{R}{a} \tag{24}$$

for this condition.

In the special case where $V_a = V_b$ (for instance both conductors at ground potential) one finds as the condition for $V' = 0$ at $r = R$:

$$2 \ln\frac{b}{R} = \ln\frac{b}{a} \text{ or } \frac{R}{a} = \frac{b}{R} \tag{25}$$

Thus, for a major ring radius R = 5 cm and a radius b = 20 cm of the outer conductor, a conducting rod of 2a = 2.5 cm diameter is needed inside the ring to make $V'(R) = 0$. If $(b/R) > (R/a)$ and $V_b = 0$, one needs, according to Eq. (22), a negative potential $V_a < 0$ on the inner conductor in order to shift the potential minimum to $r = R$.

In ERA experiments performed so far, the inner conductor is absent and $V_b = 0$. For this case, the constant $C_1 = 0$ in Eqs. (5) to (11). The potential minimum, V_o, is then located at the inner edge $r = r_1$ of the beam and the relations for $V(R)$ and $V''(R)$ in the thin-layer approximation are:

$$V''(R) = \frac{en}{\epsilon_o}(1 - \delta/R) \tag{26}$$

$$V'(R) = \frac{en\,\delta}{\epsilon_o} \qquad (27)$$

$$V(R) = V_o + \frac{en\,\delta^2}{\epsilon_o}, \text{ where } V_o = -\frac{2\,en}{\epsilon_o} R\,\delta \ln\frac{b}{R} \qquad (28)$$

Magnetic Field

In the case of a long sheet beam (L >R), we can assume that the field near the midplane has only an axial component and is uniform. The self field inside the beam ($a < r < r_1$), B_1, is opposed to the applied external field, the self field B_2 outside of the beam ($r_2 < r < b$) is in the same direction as the applied field.

By application of Ampere's Circuital Law and flux conservation one finds for the field across the beam ($r_1 < r < r_2$):

$$B(r) = -B_1 + (B_1 + B_2)\,(r^2 - r_1^2)/(r_2^2 - r_1^2), \qquad (29)$$

where

$$B_1 = \mu_o I_o (b^2 - R^2 - \delta^2)/(b^2 - a^2) \qquad (30)$$

$$B_2 = \mu_o I_o (R^2 + \delta^2 - a^2)/(b^2 - a^2) \qquad (31)$$

$$I_o = -\frac{I}{L} = -\frac{eN_o\beta c}{2\pi RL} = \text{current per unit length} \qquad (32)$$

In the thin-layer approximation, one obtains:

$$B(R) = -\frac{\mu_o I_o}{2}(a^2 + b^2)/(b^2 - a^2) - 2R^2/(b^2 - a^2) \qquad (33)$$

and for the gradient $\frac{dB}{dr}$ at r = R

$$\frac{dB}{dr}\Big/_{r=R} = B'(R) = \frac{\mu_o I_o}{2} = \frac{\mu_o eN_e\beta c}{4\pi RL\delta} \qquad (34)$$

Radial Oscillation Frequency

The radial oscillation frequency, ν_r, for a charged particle in a general axisymmetric ExB field, is given by the equation[3],

$$\nu_r^2 = 1 + \frac{E_r^2(1-\beta^2)}{(E_r + \beta cB_z)^2} + \frac{E_r}{E_r + \beta cB_z} + \frac{R}{E_r + \beta cB_z}\left[\frac{\partial E_r}{\partial r} + \beta c\,\frac{\partial B_z}{\partial r}\right] \qquad (35)$$

where

$E_r = -V'$, B_z, $\frac{\partial E_r}{\partial r} = -V''$, and $\frac{\partial B_z}{\partial r} = B'$ denote the values of the fields and the gradients at the equilibrium orbit (r = R). By substituting the results of the previous calculations into Eq. (35), one obtains the radial oscillation frequency ν_r for each case. Note that B_z is the sum of the self field, Eq. (33), and the applied external field, B_a, which, for our situation, is assumed to be uniform.

It is convenient to define a total guide field, B_g, by

$$\beta cB_g = E_r + \beta cB_z \qquad (36)$$

The force-equilibrium condition for electrons then may be written as:

$$\gamma m_o\,\beta c = eRB_g, \qquad (37)$$

where γm_o is the relativistic electron mass.

Furthermore, we introduce the parameters

$$G = 2\,\frac{\ln(b/R)}{\ln(b/a)} - 1 \qquad (38)$$

$$\nu = \frac{e^2 N_e}{4\pi\epsilon_o m_o c^2 L} \qquad (39)$$

The following expressions for ν_r^2 are then obtained:

1. Inner conductor present, $V_a = V_b$:

$$\nu_r^2 = 1 + \frac{\nu^2(1-f)^2G^2}{\gamma^4\beta^4} + \frac{\nu}{\gamma\beta^2}\left(f - \frac{1}{\gamma^2}\right)\frac{R}{\delta} \qquad (40)$$

2. $E_r = 0$ at r = R, $V_a = V_b$, $\frac{R}{a} = \frac{b}{R}$:

$$\nu_r^2 = 1 + \frac{\nu}{\gamma\beta^2}\left(f - \frac{1}{\gamma^2}\right)\frac{R}{\delta}. \qquad (41)$$

For a fully neutralized beam (f= 1) this becomes

$$\nu_r^2 = 1 + \frac{\nu}{\gamma}\,\frac{R}{\delta}, \qquad (42)$$

which is in agreement with the Lawson's result[5].

References

1. I.N. Ivanov et al., JINR Report P9-4132 (1968); unpublished.
2. L.J. Laslett, ERAN-30 (1969) and Sequel to ERAN-30 (1972), L.R.L., Berkeley; unpublished.
3. M. Reiser, Technical Report IPP 0/14 (July 1972), Garching, Germany; to be published in 'Particle Accelerators."
4. D.G. Koshkarev and P.R. Zenkevich, Particle Accelerators 3, 1 (1972).
5. J.D. Lawson, Tech. Report, Dept. Phys. & Astron. Univ. of Maryland, 1972; unpublished.

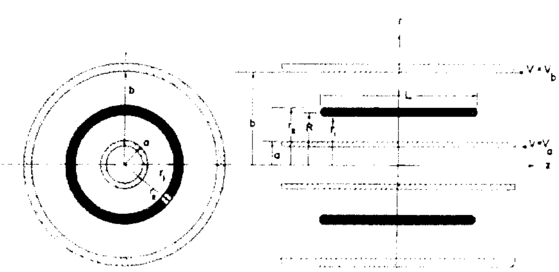

Fig. 1 Geometry of hollow beam