CALCULATION OF PARTICLE TRAJECTORIES IN AN ION LINAC WITH SELF-FOCUSING ACCELERATING GAPS HAVING QUADRUPOLAR SYMMETRY

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Summary

In linear accelerators with drift tubes the bunching of particles around the synchronous one is obtained at the expense of focusing. Nevertheless, it is possible to ensure simultaneous radial and axial stabilities if, in the accelerating gaps, we abandon the cylindrical symmetry.

If the potential function in a gap between two drift-tubes has quadrupolar symmetry, it is possible, with certain assumptions about the gap geometry, to expand this function as a double Fourier series. Thus we are able to calculate the transfer matrix from one drift tube to the next.

Formulae giving axial and radial R.F. impulses, obtained in the thin-lens approximation for the quadrupole symmetry planes are easily applicable. The effect of the R.F. is due to "circular" and "quadrupolar" impulses; the former is independent of the meridian plane, whereas the sign of the latter changes from one quadrupolar symmetry plane to the other.

The radial and axial motions of the particles cease to be independent. The chosen potential function enables us to calculate the second order transfer elements containing the coupling terms between the two degrees of freedom.

Introduction

In linear accelerators with drift tubes, the bunching of particles about the synchronous particle is achieved at the expense of the focusing. This difficultly can, under certain conditions, be surmounted if the rotational symmetry of the drift tubes is abandoned.

This possibility was first noticed by VLADIMIRSKII[1], and has recently been reconsidered by LAPOSTOLLE[2]. We propose to study the equations of motion of the ions within certain structures which possess local quadrupole symmetry. We shall consider accelerators with drift tubes, of the Sloan-Lawrence type, for which this focusing system seems to be the most promising; we shall limit the discussion to the non-relativistic case and to begin with, we neglect space charge.

The potential in the gaps

Since the operating frequencies of ion accelerators are low, the magnetic field associated with the electric field can legitimately be neglected; for the same reason, the latter can be calculated within any gap from Laplace's equation.

With Oz as the axis of the accelerator, we shall assume that the potential function $V(r,z,\theta)$ is known over the lateral surface of a cylinder, $r=a$. This assumption is convenient when considering cylindrical drift tubes whose ends have been modified to give a quadrupolar structure, rather than the systems of rectangular tubes proposed by LAPOSTOLLE[2].

With the aid of Laplace's equation, the potential in a gap can be expanded as a double Fourier series:

$$V(r,z,\theta) = \sum_n \sum_m A_{mn} \frac{I_n\left(\frac{m\pi r}{L}\right)}{I_n\left(\frac{m\pi a}{L}\right)} \cos\frac{m\pi z}{L} \cos n\theta \quad (1)$$

and we have

$$V(a,z,\theta) = \sum_n \sum_m A_{mn} \cos\frac{m\pi z}{L} \cos n\theta \quad (2)$$

Having chosen the origins of z and θ suitably, the expansion contains only cosine terms, and furthemore, n is always even (due to the quadrupole symmetry restriction on θ). In the region α, β (see fig.1), the potential distribution over the cylinder $r=a$ will depend upon θ, and the exact dependence will vary with the shape of the ends of the drift tubes.

Calculation of the parameters of the motion

a) The energy gain per gap:

If ϕ_o is the phase of the synchronous particle with respect to the RF field, the energy gain W for two gaps is given by

$$W = e\int_{-T/2}^{T/2} E_o(t)\sin(\omega t-\phi_o)\,dt \quad (3)$$

The function $E_o(t)$ is, like $E_o(z)$, symmetrical about the origin, and allowing for the synchronism relation between t and z, we obtain:

$$W = e\cos\phi_o\int_{-L}^{+L} E_o(z)\sin\frac{\pi z}{L}\,dz \quad (4)$$

while $E_o(z) = -\frac{\partial V}{\partial z}$

$$=\sum_n\sum_m A_{mn}\frac{m\pi}{L}\sin\frac{m\pi z}{L}\cos n\theta\,\frac{I_n\left(\frac{m\pi r}{L}\right)}{I_n\left(\frac{m\pi a}{L}\right)} \quad (5)$$

Only terms of order m=1 remain in the expansion, therefore, and thus:

$$W = e\cos\phi_o\,\pi\sum_n A_{1n}\cos n\theta\,\frac{I_n\left(\frac{\pi r}{L}\right)}{I_n\left(\frac{\pi a}{L}\right)} \quad (6)$$

b) The radial motion in the meridian planes $\theta=0$ and $\theta=\pi/2$

The radial field E_r is given by:

$$E_r = -\frac{\partial V}{\partial r} \quad (7)$$

$$=-\sum_n\sum_m A_{mn}\frac{m\pi}{L}\cos\frac{m\pi z}{L}\cos n\theta\,\frac{I_n'\left(\frac{m\pi r}{L}\right)}{I_n\left(\frac{m\pi a}{L}\right)}$$

in which $\cos\theta=\pm 1$

If E_r is expanded up to terms of second order in r, the radial equation of motion:

$$\ddot{r} = \frac{e}{m}E_r(r,z)\sin(\omega t-\phi_o) \quad (8)$$

becomes :

$$\ddot{r} = \frac{e}{m}\sin(\omega t-\phi_o)\left[r\left(\frac{\partial E_r}{\partial r}\right)_{o,z} + \frac{r^2}{2}\left(\frac{\partial^2 E_r}{\partial r^2}\right)_{o,z}\right] \quad (9)$$

Using the properties of BESSEL functions and their successive derivatives, we see that the only non-zero terms of $(\partial Er/\partial r)_{o,z}$ are obtained when n=0 and n=2. Similarly, the second derivative, $(\partial^2 E_r/\partial r^2)_{o,z}$ always vanishes since n is even.

The abscissa, z, of any particle will be measured with respect to that of the abscissa, z_s, of the synchronous particle, and we write $z=z_s+\varepsilon$. We can obtain the radial impulse, $\Delta\dot{r}$, from the thin lens approximation, in which the variations of r and ε during the passage across the gap are neglected:

$$\Delta\dot{r} = \frac{e}{m}r\int\left(\frac{\partial E_r}{\partial r}\right)_{0,z_s+\varepsilon}\sin(\omega t-\phi_o)\,dt \quad (10)$$

The separation between the abscissae ε, can alternatively be expressed as a phase difference, $\Delta\phi$, such that $\Delta\phi=\pi\varepsilon/L$, which leads to the equation:

$$\Delta\dot{r} = \frac{e}{m}r\int\left(\frac{\partial E_r}{\partial r}\right)_{0,z_s}\sin(\omega t-(\phi_o+\Delta\phi))\,dt \quad (11)$$

The radial impulse, $\Delta\dot{r}$, consists of two terms, P_o and P_2, which correspond to n=0 (rotationnally symmetrical term) and to n=2 (quadrupolar term), respectively. The value of P_o can be obtained easily by using the symmetry properties of the field (for n=0, m is always odd). As a result of the orthogonality of the trigonometric functions, only the A_{10} term remains, and we obtain finally:

$$P_o = r\frac{e}{m}\frac{\pi}{\omega}\frac{\pi^2}{L^2}A_{10}\frac{\sin(\phi_o+\Delta\phi)}{4}\,\frac{1}{I_o\left(\frac{\pi a}{L}\right)} \quad (12)$$

We notice that for n=2 and m=0, the ratio $I_n\left(\frac{m\pi r}{L}\right)/I_n\left(\frac{m\pi a}{L}\right)$ is equal to r^2/a^2, so that the corresponding term in the expansion of $\partial Er/\partial r$ is equal to $2A_{o2}/a^2$. From this and the symmetry properties (for n=2, m is even), we obtain the following expression for the quadrupolar impulse, P_2:

$$P_2 = \pm r\frac{e}{m}\cos(\phi_o+\Delta\phi)\times \quad (13)$$

$$\times\int_0^{\frac{T}{2}}\left(\frac{2A_{o2}}{a^2}+\sum_m A_{m2}\frac{m^2\pi^2}{4L^2}\frac{\cos m\omega t}{I_2\left(\frac{m\pi a}{L}\right)}\right)\sin\omega t\,dt$$

We now replace $\sin\omega t$ in the interval 0, T/2 by its expansion:

$$\sin\omega t = \frac{2}{\pi}\left(1-\frac{2}{3}\cos 2\omega t-\frac{2}{15}\cos 4\omega t\ldots\right) \quad (14)$$

The non-zero terms of the so obtained series decrease effectively as $1/m^4$. To an excellent approximation, therefore, we can retain only the first two terms.

If, furthermore, we replace $I_2(z)$ by $z^2/8$ we obtain:

$$P_2 \simeq \pm \frac{e\,r}{m} \cos(\phi_o + \Delta\phi) \frac{2T}{\pi a^2} \left(A_{02} - \frac{A_{22}}{3} \right) \tag{15}$$

Two types of impulse result: P_o (a "circular" impulse), which always has a defocusing effect (as have cylindrical drift tubes) and P_2 (a "quadrupolar" impulse) which is characteristic of the field asymmetry, and may be either focusing or defocusing in each gap, even though it has a net focusing effect over two successive gaps.

c) The longitudinal motion for $\theta=0$ and $\theta=\pi/2$

In terms of the variable $\varepsilon = z - z_s$ the longitudinal equation of motion,

$$\ddot{z} = \frac{e}{m} E_z(z,r) \sin(\omega t - \phi_o) \tag{16}$$

becomes, to third order approximation,

$$\ddot{\varepsilon} = \frac{e}{m} \left[\varepsilon \frac{\partial E_z}{\partial z} + r \frac{\partial E_z}{\partial r} + \frac{\varepsilon^2}{2} \frac{\partial^2 E_z}{\partial z^2} + \frac{r^2}{2} \frac{\partial^2 E_z}{\partial r^2} + \varepsilon r \frac{\partial^2 E_z}{\partial r \partial z} \right] \tag{17}$$

The following remark will enable us to simplify the calculations while increasing their accuracy. We replace the first two terms of the expansion

$$\left(\frac{\partial E_z}{\partial z}\right)_{z_s,o} + r \left(\frac{\partial^2 E_z}{\partial r \partial z}\right)_{z_s,o}$$

by the function $\left(\frac{\partial E_z}{\partial z}\right)_{z_s,r}$

The longitudinal impulse, $\Delta\dot{\varepsilon}$, can again be calculated in the way described above. The result appears in the form of a sum of expressions obtained for increasing values of n. If we consider only the first two values, n=o and n=2, we obtain:

$$\Delta\dot{\varepsilon} = \frac{e}{m} \varepsilon \left[-\frac{\pi^2}{L^2} A_{10} \sin\phi_o \frac{\pi}{2\omega} \frac{I_o\left(\frac{\pi r}{L}\right)}{I_o\left(\frac{\pi a}{L}\right)} \pm \frac{r^2}{a^2} \frac{8\pi^2}{3L^2} \frac{1}{\omega} A_{22} \cos\phi_o - \frac{\varepsilon}{2} \frac{\pi^3}{L^3} \cos\phi_o \frac{\pi}{2\omega} A_{10} \frac{I_o(\pi r/L)}{I_o(\pi a/L)} \right] \tag{18}$$

$$+ \frac{e}{m} \frac{r^2}{2} \left[\frac{\pi^3}{L^3} A_{10} \frac{\pi \cos\phi_o}{2 I_o\left(\frac{\pi a}{L}\right)} \frac{1}{2\omega} \mp \frac{32}{3} \frac{\pi^3}{L^3} A_{22} \frac{\sin\phi_o}{4 I_2\left(\frac{2\pi a}{L}\right)} \right]$$

Like the radial impulse, the longitudinal impulse consists of "circular" terms, which are also found with accelerators with the classical kind of drift tubes, together with "quadrupolar" terms which are due to the asymmetry of the gaps.

For the axial motion, the quadrupolar terms are coupling terms (and their influence is always undesirable), whereas for the radial motion, it is their presence that makes focusing possible.

We notice too that the type of coupling is not the same for the two components of the motion; of the two functions ($\varepsilon = 0, r \neq 0$) and ($r=0, \varepsilon \neq 0$), only the latter is a solution of the equations of motion.

The above expressions allow us to calculate the trajectories through a self-focusing quadrupole structure with progressively greater accuracy. If, in particular, we restrict the discussion to first order terms, the components of the motion are not coupled, and the matrix formalism can be employed. The usual stability condition: $-1 < 1/2$ (Spur of the transfer matrix) $< +1$ enables us to cheek the effectiveness of the focusing system. The real trajectories are determined by taking the coupling terms into account. It is convenient to rewrite expressions (12)(15) and (18) in terms of a unit of time equal to half the period of the RF field, since only dimensionless quantities then appear. The following approximate equations are obtained (in which certain coupling terms in εr^2 with very small coefficients have been neglected:

$$\Delta\dot{r} = r \left[\frac{\pi^2}{8} A'_{10} \sin\phi_o \frac{V_m}{V} \pm \cos\phi_o \frac{4\pi e}{\omega^2 m} \frac{V_m}{a^2} \left(A'_{02} - \frac{A'_{22}}{3} \right) + \frac{\varepsilon}{L} \left(\frac{\pi^3}{8} A'_{10} \cos\phi_o \frac{V_m}{V} \mp \sin\phi_o \frac{4\pi^2 e}{\omega^2 m} \frac{V_m}{a^2} \left(A'_{02} - \frac{A'_{22}}{3} \right) \right) \right] \tag{19}$$

$$\Delta\dot{\varepsilon} = \varepsilon \left[-\frac{\pi^2}{4} A'_{10} \sin\phi_o \frac{V_m}{V} \pm \frac{r^2}{a^2} \frac{4\pi}{3} \frac{V_m}{V} A'_{22} \cos\phi_o - \frac{\varepsilon}{L} \cos\phi_o \frac{\pi^3}{8} A'_{10} \frac{V_m}{V} \right] + \frac{r^2}{L} \left[\cos\phi_o \frac{V_m}{V} A'_{10} \frac{\pi^3}{16} \mp \sin\phi_o \frac{e}{m} \frac{8\pi^2}{3\omega^2 a^2} V_m A'_{22} \right]$$

where Vm denotes the maximum potential in the gap,and V,the mean energy of the particle expressed in electron volts.

A special case

We have applied the foregoing results to a particular potential distribution which is well-suited to represent drift tubes elongated with"fingers"(3). The function selected to represent the potential over the cylinder r=a (a being the internal radius of the drift tubes) is shown in fig.2.

To calculate the three coefficients characteristic of the field geometry,we analyse V(a,z,θ) as a Fourier series,and obtain the following values:

$$A'_{10} = \frac{2}{\pi}\,\frac{\sin\frac{\pi}{2}\frac{g}{L}}{\frac{\pi}{2}\frac{g}{L}}\cos\pi\frac{h}{L}$$

$$A'_{22} = -\frac{1}{\pi}\sin\frac{2\pi h}{L}\,\frac{\sin\pi\frac{g}{L}}{\pi\frac{g}{L}} \qquad (20)$$

$$A'_{02} = \frac{h}{L}$$

The parameters h and g are defined in fig.2

From these expressions,we shall be able to determine the particle trajectories in a real machine (3),making due allowance for the very considerable coupling which arises in self-focusing structures.

References

(1) VLADIMIRSKII V.V. Pribory i Tekhnika Eksperimenta n°3,1956 p.35

(2) LAPOSTOLLE P. Comptes Rendus Acad.Sci. Paris 256 1953 p.5294

(3) BOUSSARD D. and A.SEPTIER Particle Accelerator conference.Washington. March 1965 FF19

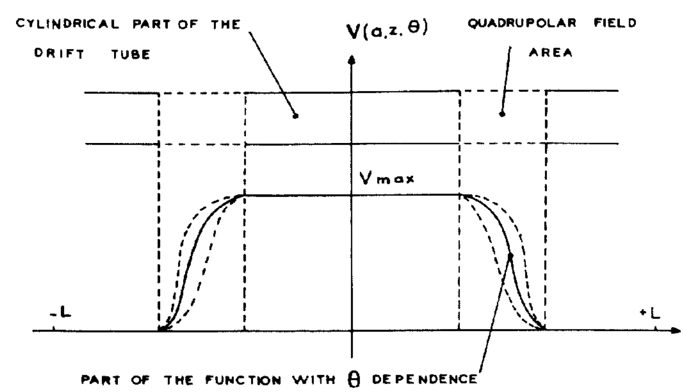

Fig.1 Potential function over the lateral surface of the cylinder r=a

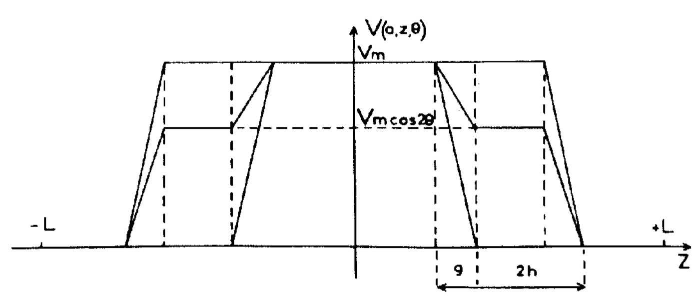

Fig.2 Potential function V(a,z,0) corresponding to drift tubes with "fingers"