

A STUDY OF FIELD DISTRIBUTION AND BEAM LOADING  
IN PROTON LINACS AT HIGH ENERGIES\*

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1. Introduction

The periodically-loaded waveguide structure being considered for proton linacs is of the standing-wave type in contrast to the traveling-wave type used for electron linacs. One reason for this choice is to enable one to control the accelerating field in the presence of beam loading. However, this control is not straightforward and, in particular, the effect of beam loading is not as simple as one guesses from the behavior of a single cavity. For example, for the  $\pi$ -mode structure which is often chosen because of its optimum shunt impedance, there is a large phase shift and field variation due to wall losses and beam loading. In this paper the effects of beam loading in a cavity with multiple cells is studied by the methods of normal-mode analysis used by the author<sup>1</sup> in an earlier study in which the details of cell structure were neglected.

We start from the cavity field equations of Slater<sup>2</sup> in which the actual field is analyzed in normal modes. The equation for the  $n^{\text{th}}$  normal-mode component of electric field,  $\vec{E}_n$ , is

$$\frac{dt}{dt^2} \int_V \vec{E}_n \cdot \vec{E}_n \, dv + \omega_n^2 \int_V \vec{E}_n \cdot \vec{E}_n \, dv = - \frac{1}{\epsilon} \frac{d}{dt} \int_V \vec{J} \cdot \vec{E}_n \, dv - \frac{\omega_n}{\sqrt{\epsilon\mu}} \int_S (\vec{E} \times \vec{H}_n) \cdot \vec{n} \, ds \quad (1.1)$$

where  $\vec{E}_n$  and  $\vec{H}_n$  are the normal electric and magnetic fields of frequency,  $\omega_n$ , which exist in an ideal cavity with a closed, perfectly conducting boundary. The surface integral in (1.1) is performed over the non-ideal surfaces divided into two parts, one part being the coupling holes and the other the lossy metallic walls. If one applies (1.1) to each cell of a cavity composed of equivalent cells, one gets the dispersion relation for the accelerating guide. For a lossless guide without beam loading, the field coefficient,

$$\int_V \vec{E}_n \cdot \vec{E}_n \, dv,$$

satisfies the equation of an oscillator having the proper frequency  $\omega_n$  and being driven by the surface integral at the coupling holes. Using Floquet's theorem for this driving force, the dispersion relation is given as<sup>3,4</sup>

$$(\omega_0^2 - \omega^2) = B (1 - \cos kL_0), \quad (1.2)$$

where  $\omega_0$  is the proper frequency of the zero mode which is assumed to be the dominant mode in each

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cell,  $k$  the real propagation constant,  $L_0$  the cell length, and  $B$  the coupling coefficient which equals the product of bandwidth and average frequency of the passband.

2. Effect of Wall Losses

Wall losses of the cavity are taken into account by writing the surface integral as

$$\frac{\omega_n}{\sqrt{\epsilon\mu}} \int_{S \text{ wall}} (\vec{E} \times \vec{H}_n) \cdot \vec{n} \, ds = \omega (1 + j) \frac{1}{Q_0} \frac{d}{dt} \int_V \vec{E}_n \cdot \vec{E}_n \, dv, \quad (2.1)$$

which is similar to the standard evaluation of  $Q_0$  from wall losses. Inserting (2.1) into (1.1), we have the equation of a damped oscillator. The dispersion relation becomes

$$\omega_0^2 - \omega^2 (1 + \frac{j}{Q_0}) + j\omega^2 \frac{1}{Q_0} = B (1 - \cos k' L_0), \quad (2.2)$$

where the coupling coefficient is assumed to be unaffected by wall losses. The propagation constant  $k'$  is now complex.

Now consider a cavity consisting of  $N-1$  uniform cells and two end cells of half length extending along the  $z$  axis. The axial field of the  $n^{\text{th}}$  normal mode of the cavity will be written as

$$E_n(m) = E_{n0} \cos k_n (N-m) L_0, \quad k_n L_0 = 0, \pi/N, \dots, \pi/N-1, \text{ and } \pi \quad (2.3)$$

in the  $m^{\text{th}}$  cell in which the field is assumed to be constant, or essentially the zero mode ( $m = 0, 1, 2, \dots$  referring to the input end).<sup>5</sup> In an actual cavity, if the power is fed into the cavity from the input end at  $z = 0$  and terminated at the completely reflecting end at  $z = L = NL_0$ , the axial field will be expressed as

$$E(m) = E_e e^{j\omega_n t} \cos k'_n (N-m) L_0 \quad (2.4)$$

Neglecting the coupling  $Q$  between the cavity and external circuits, the resonance condition gives

$$\text{Re} (\cos k'_n L_0) = \cos k_n L_0 \quad (2.5a)$$

$$I_m (\cos k'_n L_0) = - \frac{\omega'^2}{BQ_0} \equiv - \Delta_n \quad (2.5b)$$

and

$$\omega'_n \approx \omega_n \left(1 - \frac{1}{2Q_o}\right) . \quad (2.5c)$$

Thus for the  $\pi$  mode,

$$\cos k'_\pi L_o = - (1 + j\Delta_\pi) . \quad (2.6)$$

If terms of  $\Delta_\pi$  above the second order are neglected, the field in the  $m^{\text{th}}$  cell is given as

$$E_\pi(m) = (-1)^{N-m} \left[ 1 - \frac{(N-m)^4}{6} \Delta_\pi^2 + j (N-m)^2 \Delta_\pi \right] E_e e^{j\omega'_n t} \quad (2.7)$$

The phase shift and amplitude change, referring to the last cell ( $m = N$ ) are

$$\varphi_m \approx \tan^{-1} (N-m)^2 \Delta_\pi \quad (2.8)$$

and

$$\frac{|E_\pi(m)|}{|E_\pi(N)|} \approx \sqrt{1 + \frac{2}{3} (N-m)^4 \Delta_\pi^2} \quad (2.9)$$

These equations are in accord with the result by an equivalent circuit by Nagle and Knapp<sup>6</sup> when  $N^2 \Delta_\pi \ll 1$ , whereas the present result will be applicable even when  $N^2 \Delta_\pi \leq 1$  if  $\Delta_\pi \ll 1$ . For a typical  $\pi$ -mode cavity with 30 cells,  $Q_o = 2 \times 10^4$  and a bandwidth of 8%, (2.8) and (2.9) give a total phase shift of  $35^\circ$  and an amplitude change of 12%. As shown formerly,<sup>1</sup> the amplitude change can be removed by the tuning of each cell to each different frequency, but a phase shift of the same order will still remain. Since these two effects are also produced by tuning errors and beam loading, the  $\pi$ -mode structure is not a good choice for proton linacs unless it has a very broad bandwidth.

If we take the  $\pi/2$  mode instead of the  $\pi$  mode, in a structure with even  $N$

$$\cos k'_{\pi/2} L_o = - j\Delta_{\pi/2} \quad (2.10)$$

Fields in the even-numbered cells are nearly  $\pm E_e \exp(j\omega'_n t)$  and

$$E_{\pi/2}^{\text{ev}}(m) = (-1)^{N-m/2} \left[ 1 + \frac{(N-m)^2}{2} \Delta_{\pi/2}^2 \right] E_e e^{j\omega'_n t} \quad (2.11)$$

In odd cells the fields are almost zero, and given by

$$E_{\pi/2}^{\text{od}}(m) = (-1)^{N-m+1/2} j(N-m) \Delta_{\pi/2} E_e e^{j\omega'_n t} \quad (2.12)$$

There is no phase shift from cell to cell up to the second order, and only an amplitude variation remains. The field variation between the first cell and the last cell is given by

$$\frac{|E_{\pi/2}(0)|}{|E_{\pi/2}(N)|} = 1 + \frac{N^2}{2} \Delta_{\pi/2}^2 \quad (2.13)$$

For a structure with the same  $\beta$  and length as for the 30-cell  $\pi$ -mode case cited above, even taking into account the doubling of  $N$  and the reduction in bandwidth and  $Q$  value, the maximum nonflatness will be less than 1% for the  $\pi/2$  mode as compared with 12% for the  $\pi$  mode. In terms of transmission line theory, this is, of course, due to the finite group velocity and, accordingly, the small attenuation constant ( $\alpha = \omega/v_g Q_o$ ) of this mode.

### 3. Effect of Beam Loading

The effect of beam loading is also included in (1.1) by the beam coupling integral,

$$\int_V \vec{J} \cdot \vec{E}_n dv .$$

Using the normal-mode representation of (2.3) and the periodicity of the fields we can calculate the coupling integral as

$$\int_V \vec{J} \cdot \vec{E}_n dv = E_{no} I_o L \sum_{s=-\infty}^{\infty} T_s \cos \omega_{bs} t , \quad (3.1)$$

where a tightly bunched beam passing along the axis is assumed, the mean current and the bunch separation of which are  $I_o$  and  $2\pi\nu/k_n$  ( $\nu = \text{an integer}$ ), respectively.  $\omega_{bs}$  and  $T_s$  in (3.1) are

$$\omega_{bs} = \left( k_n + \frac{2\pi s}{L_o} \right) v_o , \quad (3.2)$$

$$T_s = \frac{\sin \left( \frac{(k_n L_o + 2\pi s)}{2} \right)}{\left( \frac{k_n L_o + 2\pi s}{2} \right)} \quad (3.3)$$

The velocity of particles,  $v_o$ , is taken to be constant along the cavity.

Inserting (3.1) and (2.1) into (1.1), we find an equation describing a forced oscillation induced by the beam. In general, such a forced oscillation can be maintained at the steady state only when the proper oscillation of the cavity is in resonance with the external force. This synchronized condition is expressed as  $\omega'_n = \omega_{bo} = k_n v_o$  and leads to the field coefficient

$$\int_V \vec{E}_b \cdot \vec{E}_n dv = - \frac{E_{no} I_o L T_o \tau_n Q_o}{2 \epsilon \omega'_n} e^{j\omega'_n t} , \quad (3.4)$$

where  $\tau_n$  is 2 for the zero or  $\pi$  mode and 1 for other modes. Using a relation between the effective shunt impedance and the unloaded  $Q$  value of<sup>1,5</sup>

$$\frac{r_e}{Q_o} = \frac{E_{no}^2 L T_o^2 \tau_n}{2 \epsilon \omega'_n} , \quad (3.5)$$

the axial field induced by the beam is given as

$$E_b(m) = - E_b e^{j\omega'_n t} \cos k(N-m) L_o , \quad (3.6a)$$

$$E_b = r_e I_o / T_o \quad (3.6b)$$

which is out-of-phase with that of the beam bunch.

Then the total field in the  $m^{\text{th}}$  cell of a  $\pi$ -mode cavity is given by the addition of (3.6) to (2.6) giving

$$E_{\pi}(m) = (-1)^{N-m} \left[ \left( 1 + j(N-m)^2 \Delta_{\pi} \right) E_e - E_b e^{j\varphi_{bN}} \right] e^{j\omega' t}, \quad (3.7)$$

where  $\varphi_{bN}$  is the phase angle of the bunch in the last cell and the term proportional to  $\Delta_{\pi}^2$  is neglected. The vector diagram of the field composition is shown in Fig. 1, and the phase angle between the beam and the field is given by

$$\varphi_{bm} = -\varphi_{bN} + \tan^{-1} \frac{(N-m)^2 \Delta_{\pi} E_e - E_b \sin \varphi_{bN}}{E_e - E_b \cos \varphi_{bN}}. \quad (3.8)$$

If we introduce the beam  $Q$  value,  $Q_b$ , and phase constant,  $\alpha_b$ , as

$$\frac{1}{Q_b} = \frac{1}{Q_o} \frac{E_b \cos \varphi_{bN}}{E_e - E_b \cos \varphi_{bN}}, \quad (3.9a)$$

$$\alpha_b = \tan \varphi_{bN} \quad (3.9b)$$

(3.8) becomes

$$\varphi_{bm} = -\varphi_{bN} + \tan^{-1} \left[ (N-m)^2 \Delta_{\pi}' - \frac{\alpha_b Q_o}{Q_b} \right] \quad (3.10)$$

$\Delta_{\pi}'$  is given by (2.5b), with  $1/Q_o$  replaced by  $1/Q_t = 1/Q_o + 1/Q_b$ . Comparing with (2.8) we find two effects due to the beam; one is the additional loss by beam loading which results in an amplification of  $\Delta_{\pi}$ , and the other is due to the reactive component of the beam loading. The latter is proportional to  $\alpha_b$  and independent of  $m$ , so that it will be eliminated by a shift,  $\Delta\omega$ , of the operating frequency,

$$\frac{\Delta\omega}{\omega_{\pi}} \approx \frac{\alpha_b}{2Q_b}. \quad (3.11)$$

As we can see from Fig. 1, the best choice of the particle phase in a  $\pi$ -mode cavity will be  $\varphi_{bN} \approx \alpha_b \approx 0$  ( $\varphi_{bN} \ll 0$  for phase stability), and this term should be negligible. However, the effect of beam loading on  $\Delta_{\pi}$  still remains after the tuning and the cell-by-cell phase shifts will be appreciably affected by the beam, if  $Q_b$  is of the same order as  $Q_o$  which occurs for  $I_o \sim 0.1$  A,  $r_e \sim 20$  M $\Omega$ /m and  $E_e \sim 5$  MeV/m. Similarly, an appreciable change of the field flatness in a  $\pi$ -mode cavity is expected from beam loading.

For a  $\pi/2$ -mode cavity, the total field is given by the combination of (3.6) with (2.11) and (2.12). In this case the phase shift in even-numbered cells only due to the reactive component will exist in addition to the decrease of field

amplitude by loading. Such effects are almost independent of cell number and can be eliminated by two adjustments: a small change of the operating frequency, estimated from (3.11) to be of the order of 10 kc, and an increase of the input power.

In the above considerations the coupling effect between the beam and field which causes the beam blowup instability in high-current electron accelerators was neglected. The analyses of such phenomena have been given in various reports<sup>7</sup> and will be somewhat improved by using equation (1.1) as a circuit equation for each cell. For example, in a  $\pi$ -mode cavity we find four component waves, i.e. the increased, the attenuated, and two non-attenuated waves, as compared with the usual three waves in a traveling-wave structure or a  $\pi/2$ -mode standing-wave structure. The Pierce-type non-dimensional coupling constants for the  $\pi$  and  $\pi/2$  modes will be related by

$$C_{\pi}^4 / C_{\pi/2}^3 \approx \frac{1}{4\pi}. \quad (3.12)$$

For typical values of the parameters,  $C_{\pi}$  will be about twice  $C_{\pi/2}$ . However, because of the small attenuation constant for the  $\pi/2$  mode, the starting current for the blowup effect may be smaller for the  $\pi/2$  than for the  $\pi$  mode. At any rate, in a proton linac both starting currents will be of the order of 1 A and the difference is not as important as in the electron case.

In conclusion, from the consideration of wall losses and beam loading given above, and for the tuning errors discussed in the Appendix, the  $\pi/2$  mode is superior to the  $\pi$  mode. However, from the consideration of shunt impedance the  $\pi/2$  mode is believed to be inferior. If one still uses the  $\pi$  mode because of its high shunt impedance, then it will be necessary to find a broad bandwidth structure. A combined  $\pi/2$ - and  $\pi$ -mode structure with a multiperiodicity as proposed by Giordano<sup>8</sup> is of interest in this connection.

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APPENDIX  
Effect of Tuning Errors

The effect of tuning errors in each cell on the field distribution has already been studied in some aspects of proton linac studies.<sup>5,6</sup> This work is done to show clearly the different behavior between the  $\pi$  and  $\pi/2$  modes.

By analogy with Panofsky's method for an Alvarez cavity, the tuning errors along the cavity are written as a Fourier series,

$$\omega'_n(z) = \omega'_n \left( 1 + \sum_{r=1}^{\infty} p_r \cos \frac{r\pi z}{L} \right) \quad (A.1)$$

For a general study, we consider the field variation corresponding to these errors as an expansion equation in normal modes.

Assuming the z-dependence of the axial field in a normal mode as  $\cos n\pi z/L$ , the perturbed field is given by

$$E_n(z) = E_0 \left( \cos \frac{n\pi z}{L} + \sum_{s=-\infty}^{\infty} \eta_s \cos \frac{s\pi z}{L} \right), \quad (A.2)$$

where the unperturbed field is  $E_0 \cos n\pi z/L$ .

By a standard perturbation method with the consideration of orthogonality relations and the normalization condition, one can get

$$\eta_s = \frac{p_r \omega_n^2}{\omega_n^2 - \omega_s^2} \quad (A.3)$$

where  $r(>0)$  takes  $(n-s)$  or  $-(n-s)$  values depending on the sign of  $(n-s)$ . Inserting (A.3) into (A.2), we can write

$$E_n(z) = E_0 \cos \frac{n\pi z}{L} \left[ 1 + \sum_{r=1}^{\infty} \epsilon_r \cos \frac{r\pi z}{L} \right] \quad (A.4)$$

with

$$\epsilon_r = p_r \omega_n^2 \left[ \frac{1}{\omega_n^2 - \omega_{n+r}^2} + \frac{1}{\omega_n^2 - \omega_{n-r}^2} \right] \quad (A.5)$$

These equations accord with Panofsky's formula when  $n = 0$ , and are also applicable to any perturbed normal mode. For the  $\pi$  mode, using the dispersion equation of (1.2), one gets the amplification factor,

$$\frac{\epsilon_r}{p_r} = \frac{4 N^2 \omega_n^2}{\pi^2 r^2 B} \quad (A.6)$$

on condition that  $r \ll N$ . This expression shows a fairly good agreement with the experiments by Giordano,\* and the calculated amplification factor reaches  $4.5 \times 10^3$  for a 30-cell structure with a bandwidth of 8%. In contrast, in the  $\pi/2$  mode, for a symmetric dispersion curve,  $\omega_{n+r} - \omega_n = -(\omega_{n-r} - \omega_n)$  in (A.5), so that  $\epsilon_r$  is cancelled in the first order and only a small effect due to the tuning error is to be expected.

\*Private communication.

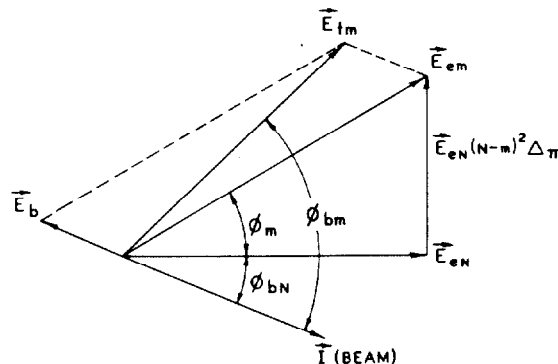


Figure 1. Vector diagram of the field in the  $m^{\text{th}}$  cell of a  $\pi$ -mode cavity.