NONLINEAR STABILITY OF INTENSE MISMATCHED BEAMS IN A UNIFORM FOCUSING FIELD*

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Abstract

We investigate the nonlinear coupling between breathing and quadrupole-like oscillations in the dynamics of intense beams propagating in a uniform magnetic focusing field. It is shown that finite amplitude breathing oscillations of an initially round beam may destabilize quadrupole-like oscillations, heavily affecting stability and the shape of the beam. This is a potential mechanics for beam particle loss in such systems.

INTRODUCTION

A key issue to be overcome in the development of high-intensity accelerators and vacuum electronic devices is the prevention of particle beam losses. In order to achieve that, a crucial ingredient is a better understanding of the beam transport stability. Many studies have been made on the linear stability of uniform and periodically focused beams [1, 2, 3, 4]. They detected the occurrence of different instability modes which compromise beam transport for certain parameters of the system. Of particular relevance for axisymmetric solenoidal focusing is the breathing mode that induces increasing-amplitude axisymmetric oscillations of the beam envelope around its matched (equilibrium) value; and the quadrulope-like mode that induces elliptic oscillations of the beam, breaking its symmetry [2]. Nonlinear stability analysis were also performed but restricted to axisymmetric beams [5].

In this paper, we analyze the nonlinear stability of beams taking into consideration nonaxisymmetric effects. We investigate the nonlinear coupling between breathing modes and quadrupole-like modes based on envelope equations. It is shown that finite amplitude breathing oscillations caused by some sort of mismatch, such as the one induced by current oscillations in microwave sources [6], may drive unstable quadrupole-like oscillations for an initially *quasi*-axisymmetric beam. In this case, the beam starts developing an elliptical shape with a consequent increase in its size along one direction. This may induce beam losses, which are enhanced if conducting wall effects are taken into account [7], and may also induce a detuning in the wave-beam interaction in high-power microwave tubes.

THE MODEL

We consider a thin, continuous beam propagating with average axial velocity $\beta_b c \hat{\mathbf{e}}_z$ along an uniform solenoidal magnetic focusing field $\mathbf{B}(\mathbf{x}) = B_z \hat{\mathbf{e}}_z$, where c is the speed of light in *vacuo*. Although we restrict our analysis to an uniform focusing, the results are expected to be valid for periodic focusing too, as long as smooth-beam approximations are valid [4]. Since we are dealing with solenoidal focusing, it is convenient to work in the Larmor frame of reference [4], which rotates with respect to the laboratory frame with angular velocity $\Omega_L = qB_z/2\gamma_b mc$, where q, m and $\gamma_b = (1 - \beta_b^2)^{-1/2}$ are, respectively, the charge, mass and relativistic factor of the beam particles. The transverse cross section of the beam is assumed to be an ellipsis centered at $x = 0 = y$ which rotates with angular velocity Ω_L such that its semi-axes are parallel to *fixed* $x-$ and y-axes of the Larmor frame. In the paraxial approximation the equations that dictate the beam envelope evolution are

$$r_x'' + \sigma_0^2\, r_x - \frac{2K}{r_x + r_y} - \frac{\epsilon^2}{r_x^3} = 0, \qquad (1)$$

$$r_y'' + \sigma_0^2\, r_y - \frac{2K}{r_x + r_y} - \frac{\epsilon^2}{r_y^3} = 0, \qquad (2)$$

where $s = z$ is the propagation distance, the prime denotes derivative with respect to s, r_x and r_y are the ellipsis semi-axes radii, $\sigma_0 = qB_z/2\gamma_b\beta_b mc^2$ is the vacuum phase advance per unit axial length, $K = 2q^2 N_b/\gamma_b^3\beta_b^2 mc^2$ is the beam perveance, N_b is the number of particles per unit axial length, and ϵ is the unnormalized emittance of the beam. There is a particular solution of the envelope equations (1) and (2) for which $r_x = r_y = r_0 = $ const. This corresponds to the so called matched solution for which a circular beam of radius $r_0 = [K + (K^2 + 4\sigma_0^2\epsilon^2)^{1/2}]^{1/2}/(2\sigma_0^2)^{1/2}$ preserves its size throughout the transport. Linear stability calculations show that small amplitude oscillations around r_0 are always stable [2]. Our purpose here is to investigate what happens when finite amplitude oscillations are taken into consideration.

NONLINEAR STABILITY ANALYSIS

We first note that Eqs. (1) and (2) can be derived from a Hamiltonian formalism

$$H = H_x(r_x, p_x) + H_y(r_y, p_y) - 2K \log(r_x + r_y), \quad (3)$$

where

$$H_\xi = \frac{p_\xi^2}{2} + \frac{\sigma_0^2\, r_\xi^2}{2} + \frac{\epsilon^2}{2r_\xi^2}, \qquad (4)$$

$$r'_\xi = \partial H/\partial p_\xi = p_\xi, \ p'_\xi = -\partial H/\partial r_\xi, \qquad (5)$$

$\xi = x, y$. It is clear from Eq. (3) that for negligible space-charge ($K \to 0$), r_x and r_y dynamics are decoupled and nonlinear instabilities are absent.

We are interested in the stability of axisymmetric beams with $r_x(s) \approx r_y(s)$ as they undergo finite amplitude breathing oscillations around r_0. It is thus convenient to introduce new canonical variables defined as

$$X_\pm = \frac{r_x \pm r_y}{\sqrt{2}}, \ P_\pm = \frac{p_x \pm p_y}{\sqrt{2}} \qquad (6)$$

Note that X_+ and P_+ are sensitive to symmetric oscillations where r_x and r_y oscillate in phase – breathing modes; on the other hand, X_- and P_- are sensitive to antisymmetric oscillations where r_x and r_y oscillate with opposite phases – quadrupole-like modes. The dynamics is then dictated by the following hamiltonian

$$H = H_+(X_+, P_+) + H_-(X_-, P_-) + H_C(X_+, X_-), \ (7)$$

where

$$H_+ = \frac{P_+^2}{2} + \frac{\sigma_0^2 X_+^2}{2} - K \log X_+^2, \qquad (8)$$

$$H_- = \frac{P_-^2}{2} + \frac{\sigma_0^2 X_-^2}{2}, \qquad (9)$$

$$H_C = 2\epsilon^2 \frac{X_+^2 + X_-^2}{(X_+^2 - X_-^2)^2}. \qquad (10)$$

Observe that when emittance effects are negligible ($\epsilon \to 0$), the coupling Hamiltonian H_C vanishes. In this case, the symmetric and anti-symmetric oscillations become uncoupled and integrable, and instabilities are absent. Thus, we note that in both limiting cases – when emittance effects are negligible or when space-charge effects are negligible – the instability vanishes. In the new variables the matched solution is given by $X_+(s) = X_{+0} \equiv [K + (K^2 + 4\sigma_0^2\epsilon^2)^{1/2}]^{1/2}/\sigma_0$, $X_-(s) = P_-(s) = P_+(s) = 0$. This corresponds to the minimum *energy* H, obtained by imposing that the Hamiltonian derivatives with respect to all canonical variables vanish. In this sense, mismatched oscillations are excess energy given to the system. In general, the free energy may appear as oscillations in both symmetric and anti-symmetric degrees-of-freedom, and the nonlinear coupling given by H_C may induce exchange of energy between the two modes. Hence, an initially round beam undergoing breathing oscillations may, in principle, start developing a quadrupole-like instability, becoming elliptical.

To investigate this issue we examine the solutions obtained by numerically integrating Eqs. (1) and (2). The analysis is simplified if we normalize quantities according to

$$\sigma_0 s \to s, \ (\sigma_0/\epsilon)^{1/2} X_\pm \to X_\pm, \ K/(\sigma_0\epsilon) \to \eta. \quad (11)$$

Then, η is the only parameter in the equations and measures the relative strength of space-charge to emittance effects – $\eta \ll 1$ is an emittance-dominated beam, $\eta \gg 1$,
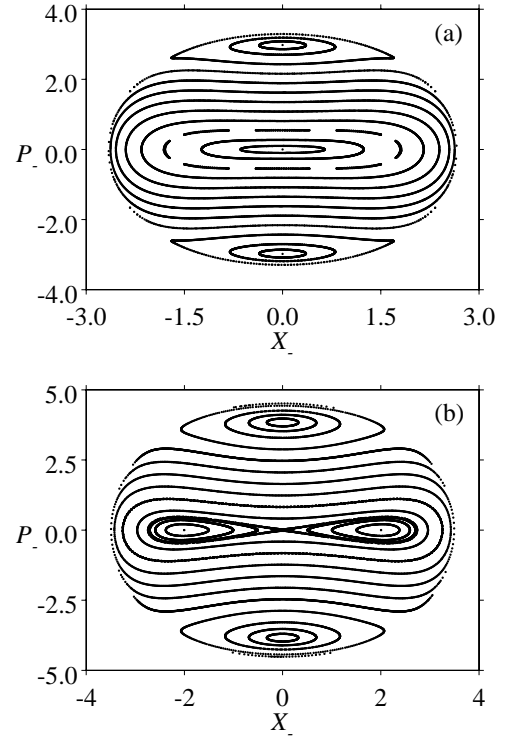

Figure 1: Poincaré Plots for the phase-space (X_-, P_-); $\eta = 3.0$, $\nu = 2.0$ (a) and $\nu = 2.4$ (b).

is an space-charge dominated beam. From what was discussed, both limits $\eta \to 0$ and $\eta \to \infty$ are integrable and stable. We begin by analyzing the $(X_+, P_+; X_-, P_-)$ phase-space. Because this is a two-degrees of freedom system, Poincaré Plots come in order. We choose to plot (X_-, P_-) each time X_+ is maximum, which is sufficient to ensure uniqueness of trajectories in our plots. In Fig. 1, we present phase-space plots obtained for $\eta = 3.0$ and different values of the mismatch amplitude ν. In the plots the axisymmetric solution corresponds to a fixed point at the origin $X_- = P_- = 0$. In panel (a), for $\nu = 2.0$, the phase-space presents some nonlinear features, such as resonant islands. Notwithstanding, the resonances are far from the fixed point and does not compromise its stability because nearby trajectories just rotate around it with no increase in X_- and P_- amplitudes. However, if we increase the mismatch amplitude to $\nu = 2.4$, as in panel (b), we notice that the axisymmetric solution becomes unstable and any small ellipticity of the beam will grow to large X_-. This is illustraded in Fig. 2, where the envelope evolution is presented for a case whith the same parameters as in Fig. 3(b) and an initial ellipticity of the beam given by $r_x(0)/r_y(0) = 1.01$. It is clear that a strong quadrupole instability is going on because the ratio r_x/r_y quickly grows up to 3.13 after just 7 mismatched oscillations of the beam at $s = 25.4$. Note, as well, that the size of the beam along the x direction increases more than 30% of its initial value due to the instability.
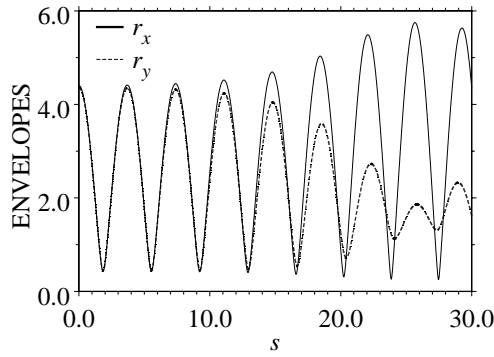
Figure 2: Envelope evolution $r_x(s)$, $r_y(s)$ for an unstable case. The parameters are the same as in Fig. 1(b) with $r_x(0)/r_y(0) = 1.01$.

Figure 3: Threshold values of the mismatch amplitude for the onset of quadrupole-like instability, ν_{th} vs. η.

In qualitative agreement with the model discussed previously, the phase-space analysis indicates that there is a threshold mismatch amplitude ν_{th} above which instability takes place. To determine with more accuracy ν_{th} and how it varies with η, we adopt the following procedure. For a given η, we numerically integrate over a long propagation length s_f the coupled envelope equations for initial conditions of the form $X_+(0) = \nu X_{+0}$, $P_+(0) = 0$, and $J_0 \equiv [X_-^2(0) + P_-^2(0)]/2 \ll 1$. If during the integration $J(s) \equiv [X_-^2(s) + P_-^2(s)]/2$ exceeds J_0 by a large factor λ ($J(s) > \lambda J_0$) we consider the solution to be unstable; if not, the solution is considered stable. By ratcheting up ν from 1 we determine $\nu_{th}(\eta)$ as the minimum value for which the solution is unstable. Specifically, we take $s_f = 100$, $J_0 = 10^{-3}$, and $\lambda = 100$. The results are presented in Fig. 3 by the circles connected by a solid curve. ν_{th} increases in both limits $\eta \to 0$ and $\eta \to \infty$, as expected, presenting a minimum $\nu_{th} \approx 1.96$ close to $\eta = 1$. For $\eta < 1$, it steeply increases; hence, the quadrupole-like instability is expected to have little effect on tenuous beams where space-charge effects are small. On the other hand, ν_{th} increases much slower for $\eta > 1$. In fact, it grows less then 30% of its minimum value as we move to very intense beams with $\eta = 20.0$. Therefore, devices that operate with space-charge dominated beams undergoing finite amplitude mismatched oscillations are likely to be affected by the instability.

CONCLUSIONS

We have analyzed the nonlinear stability of beams in magnetic focusing fields taking into consideration nonaxisymmetric effects. In particular, we investigated the nonlinear coupling between breathing modes and quadrupole-like modes using envelope equations. It was shown that finite amplitude breathing oscillations caused by some sort of mismatch, such as the one induced by current oscillations in microwave sources [6], may drive unstable quadrupole-like oscillations for an initially *quasi-*
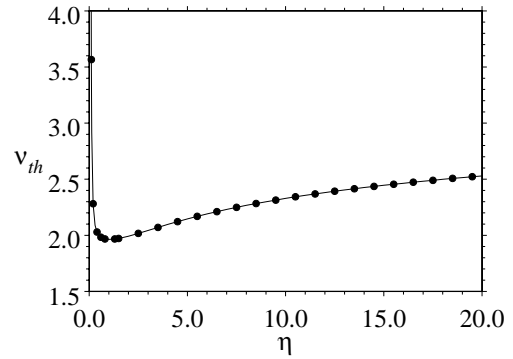
axisymmetric beam. In this case, the beam starts developing an elliptical shape with a consequent increase in its size along one direction.

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