CHAOS IN TIME-DEPENDENT SPACE-CHARGE POTENTIALS*

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Abstract

We consider a spherically symmetric, homologously breathing, space-charge-dominated beam bunch in the spirit of the particle-core model. The question we ask is: How does the time dependence influence the population of chaotic orbits? The static beam has zero chaotic orbits; the equation of particle motion is integrable up to quadrature. This is generally not true once the bunch is set into oscillation. We exemplify how the population of chaotic orbits evolves to build a halo. We also compute an integral of particle motion in this time-dependent potential, and in the process show that this computation would seem to reflect in near-real time the character of the orbital evolution, including transitions between chaos and regularity. We then introduce colored noise into the system and show how its presence modifies the dynamics.

OVERVIEW

Although detailed microscopic dynamics has been studied and characterized to a significant extent in both time-independent and time-dependent space-charge potentials [1], much remains to be learned, especially in regard to how microscopic evolution drives the macroscopic properties of the charged-particle beam. The beam can be viewed unequivocally as a collection of N interacting particle orbits. Liouville's Theorem applies rigorously to the respective $6N$-dimensional phase space, yet this fact is not especially helpful computationally. Thus, one appeals to the BBGKY hierarchy of s-particle reduced distribution functions ($s \le N$), decomposes these distribution functions into a hierarchy of correlation functions, and then appeals optimistically to time-scale arguments to truncate this hierarchy. The end result is, to lowest order, the Vlasov-Poisson-Maxwell equation, and to next order, the Landau kinetic equation [2]. In either case one has an equation for the evolution of the distribution function of particles in the 6-dimensional phase space of a single particle. Yet, to do this, one in fact sacrifices some of the physics, usually by arguing that orbits quickly "forget" their initial conditions so that their mutual interactions in effect constitute a Markov process. One major consequence of this argument is that Liouville's Theorem generally will *not* apply to the 6-dimensional phase space. For example, the kinetic equation explicitly folds in a non-Liouvillian "collision term" that enables internal free energy eventually to thermalize. From a practical point of view, this means the volume of phase space subtended by the N particles tends to grow, and hence, so does the full emittance.

In this paper, we adopt the Vlasov-Poisson picture. We take the density and potential to be coarse-grained, and we regard individual particles to move in response to this globally coarse-grained potential. In other words, we assume that the "continuum limit" is valid, although it has been shown for beams that this is not necessarily true [3]. However, our interest here is to isolate the importance of time dependence for the evolution of a beam bunch; we thus purposely avoid any other complication inherent to a real beam. As will be demonstrated, time dependence alone triggers complicated evolutionary dynamics. How the dynamics drives the macroscopic properties of the beam can readily be understood in terms of mixing, and the qualitative behavior of this mixing depends critically and unequivocally on the nature of the orbits, i.e., whether they are *regular* or *chaotic*.

Interestingly, regardless of the nature of a given orbit, one can compute an integral of its motion in the time-dependent potential. This integral has as its root the Hamiltonian, and it reduces to the total energy in the limit that the time dependence goes to zero. In the process of computing this integral, we find signs that the computation reveals sensitively the nature of the given orbit in near-real time, a "feature" that we shall elaborate.

The model that we use to illustrate the consequences of time dependence is a spherical, homologously breathing waterbag in which test particles are set into motion and tracked. This is a two-parameter family of particle-core models, one parameter being the envelope mismatch M, and the other being the space-charge-depressed tune η. Okamoto and Ikegami previously treated a cylindrical version of this model [4], and their work is partial motivation for our investigation.

One other feature that we shall consider is the effect of colored noise. This noise is unavoidable in real machines; hardware and field irregularities will self-consistently influence the space-charge potential and hence the particle orbits. This noise does work on the beam, causing emittance growth and halo formation. For a homologously breathing KV beam, the outer KAM tori are robust; modest noise strength will not break them, although they become much more fragile if internal collective modes are excited [1]. Herein we consider only a breathing waterbag; there are no internal modes. We again find the outer KAM tori to be robust; however, we also find that, for certain parameter choices, modest colored noise can nonetheless make a big difference in the macroscopic properties of the bunch, to include triggering halo formation where otherwise there would be none.

TIME DEPENDENCE AND MIXING

The breathing spherical waterbag model, once populated with test particles, constitutes a particle-core model. The corresponding equilibrium distribution

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function in the 6D ($\boldsymbol{r}$, $\boldsymbol{p}$) phase space of a single particle is $f(\boldsymbol{r},\boldsymbol{p}) \propto [H_o - H(\boldsymbol{r},\boldsymbol{p})]^{-1/2}$ for $H_o > H$, H denoting the Hamiltonian, and it is zero otherwise. The corresponding density and total potential for the bunch of charge Q are:

$$\rho = [3Q/(4\pi R^3 \nu)]\{1 - i_0[r/(R\alpha)]/i_0(\xi)\},$$

$$\Phi = \nu/(\xi R\alpha) - (1/2)(1 - R^{-3})[(\xi R\alpha)^2 - x^2] - (\mu/R) d_0[x/(R\alpha)], \qquad |x| < \xi R\alpha;$$

$$= -(1/2)[(\xi R\alpha)^2 - x^2] + \nu/|x|, \qquad |x| \geq \xi R\alpha.$$

Auxiliary quantities in these expressions are:

$$i_0(y) \equiv y^{-1}\sinh y, \; d_0(y) \equiv i_0(\xi) - i_0(y),$$

$$\alpha \equiv \left\{ \frac{(3+\xi^2) i_0(\xi) - 3\cosh\xi}{[\xi^4 + 15(2+\xi^2)] i_0(\xi) - 5(6+\xi^2)\cosh\xi} \right\}^{1/2},$$

$$\mu \equiv 3\alpha^2 / i_0(\xi),$$

$$\nu \equiv \frac{\xi\alpha^3}{i_0(\xi)}[(3+\xi^2) i_0(\xi) - 3\cosh\xi],$$

in which the intermediary parameter ξ is found from the tune η by solving

$$\frac{15}{4 i_0(\xi)} \frac{(12+5\xi^2) i_0^2(\xi) - 3\cosh\xi[3 i_0(\xi) + \cosh\xi]}{[30 + \xi^2(15+\xi^2)] i_0(\xi) - 5(6+\xi^2)\cosh\xi} = \eta^2 .$$

Time dependence is incorporated through the envelope:

$$\ddot{R} + R - \eta^2 R^{-3} - (1-\eta^2) R^{-2} = 0; \; R(0) = M, \; \dot{R}(0) = 0.$$

The equation of test-particle motion is $\ddot{x} = -\partial_x \Phi$.

There are several attractive advantages of the spherical waterbag: (1) the distribution function, the charge density, and the space-charge potential are all analytic; (2) the tune η parameterizes the family of models, so one can readily explore the range from zero space charge to the space-charge limit; (3) the equilibrium configurations are all stable, and (4) whereas all stationary configurations are integrable in keeping with spherical symmetry, chaotic orbits arise once they are made to breathe. Hence, this (deterministic) chaos is fully attributable to the time dependence alone. Thus, this family of models is ideal not only for answering basic questions such as how time dependence triggers chaoticity, but also for highlighting the efficacy of new techniques for quantifying this chaos and for analyzing nonlinear dynamics in general.

What is utterly fascinating is that one can 'watch' clumps of test particles that are initially tightly localized in phase space evolve to fill in the Poincaré surface of section (PSS); the wildly chaotic orbits quickly migrate to fill the chaotic sea, 'sticky' chaotic orbits eventually break into the chaotic sea and migrate away, while the regular orbits slowly mix to fill the regular islands. One can likewise watch chaotic particles migrate to form a halo. In fact, chaotic orbits mix exponentially through the phase space accessible to them, while regular orbits mix secularly (as a power law in time); these qualitative differences are manifest in Fig. 1.

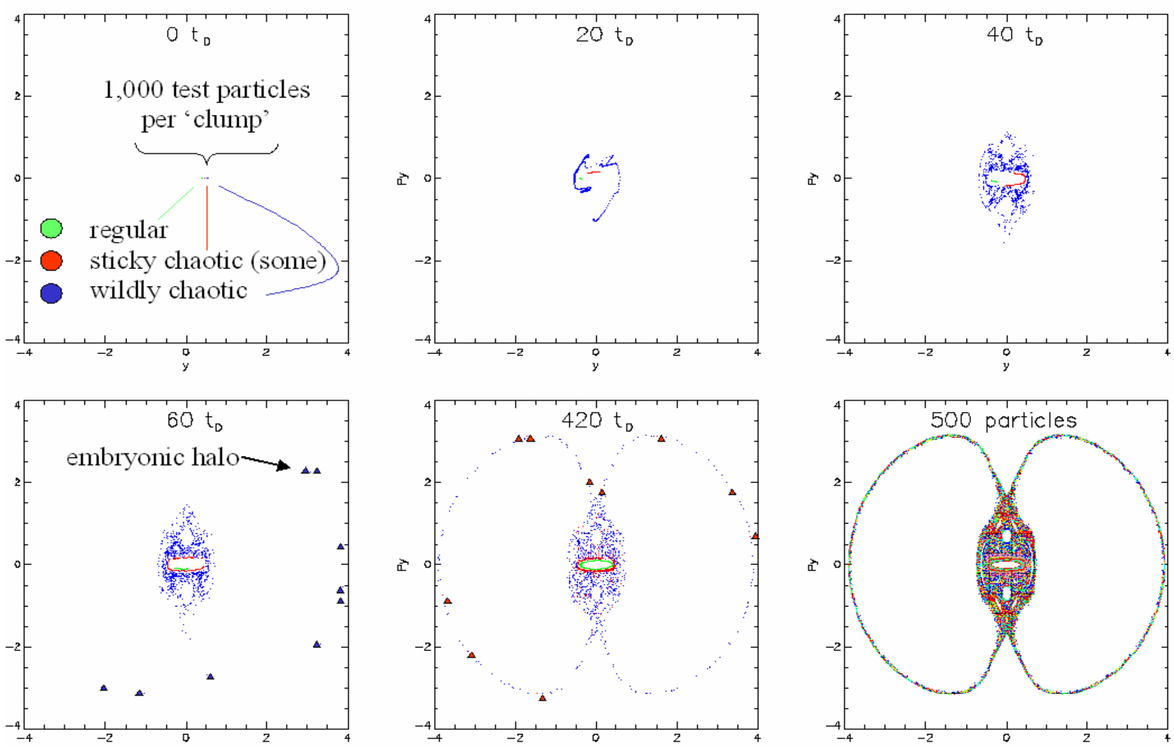

Figure 1. Stroboscopic snapshots of mixing in a spherical waterbag with M=0.5, η=0.15. The first 5 plots illustrate evolution of 3 clumps, each with 1000 initial conditions tightly localized (all with zero initial velocity) in regions of phase space known to be: wildly chaotic (blue), regular (green), and sticky chaotic (red). Hard-to-see particles are flagged. Note that in the 5th plot (t=420 t_D, i.e. 420 space-charge-depressed orbital periods) there are some red particles in the 'ears', i.e., the halo. For perspective, a 1 GeV proton linac would span ~100 t_D, i.e., this figure is meant to exemplify transient dynamics, not necessarily a real machine. The final plot shows the complete PSS computed with 500 test particles that were initially distributed according to the waterbag density profile.

AN INTEGRAL OF MOTION

We construct this integral by augmenting the technique of Struckmeier and Riedel [5]. The result for a one-dimensional potential $\Phi(x,t)$ is:

$$I = fH(x,p,t) - (\dot{f}x - b)p/2 + (\ddot{f}x - 2\dot{b})x/4 + c,$$

$$(x^2/2)\dddot{f} + (2\Phi + x\partial_x\Phi)\dot{f} + 2(\partial_t\Phi)f - x\ddot{b} - b\partial_x\Phi + \dot{c} = 0,$$

$$f(0) = 1, \dot{f}(0) = \ddot{f}(0) = 0,$$

where $b=b(t)$ and $c=c(t)$ are arbitrary functions of time that are to be chosen for convenience in solving a given problem (thereby affording freedom to define a "gauge"), and $f(t)$ is an auxiliary function of time only. To solve for the integral I computationally for the waterbag (thereby proving its conservation) requires letting the terms involving $b(t)$ add to zero and integrating a given orbit while simultaneously solving for $f(t)$ and $c(t)$. An interesting "discovery" came to us in the process. It seems that if the orbit is regular, the auxiliary functions are well-behaved (stable), but if any part of the orbit is chaotic, these functions blow up exponentially (yet the integral is still conserved and the orbit remains stable). One example of such behavior is depicted in Fig. 2. The orbit plotted there is initially chaotic, as is known by evaluating its Lyapunov exponent. Note that $f(t)$ indeed blows up from the start. Then we purposely damp the breathing, thereby causing the orbit to become regular. And, finally, the breathing is restored so that the orbit again becomes chaotic. Note that: (1) $f(t)$ continues to blow up, even during the regular interval, (2) the orbital transitions are reflected at once by way of a change of slope in the $\ln[f(t)]$ vs. t plot, and (3) the frequency content (just one frequency for the regular epoch) is clearly reflected in this plot. In view of these findings, and in view of how sensitive this function is to the detailed orbital dynamics, perhaps it can be used to distinguish, in near real-time, orbital transitions from regular to chaotic and back. This is a question to be answered in future work.

INFLUENCE OF COLORED NOISE

We incorporate colored noise in the breathing frequency ω (ω=1 here); its influence depends on its strength $\langle|\delta\omega|\rangle$ and autocorrelation time t_c. With $t_c=12t_D$, we investigated a broad range of strengths, specifically $2\cdot 10^{-4} \le \langle|\delta\omega|\rangle \le 0.05$, and found that, just as for the breathing cylindrical KV beam of a previous study [1], the outer KAM tori in the Poincaré surface of section are resilient; only very strong noise breaks them. However, a counterexample appears in Fig. 3, derived from sequential computations of 10,000 orbits initially distributed per the waterbag density, with noise randomly generated for each orbit separately. In this case, a modest noise strength $\langle|\delta\omega|\rangle$=0.01 breaks the outer tori and enables particles to leak into a halo having the same dimensions as that of Fig. 1.

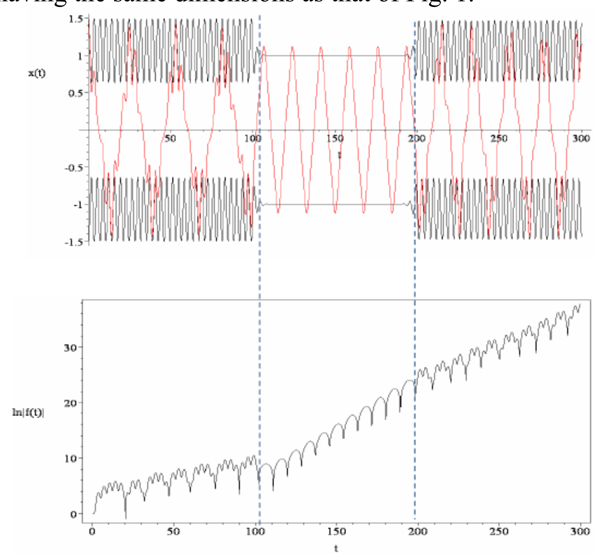

Figure 2. Transient chaos (intermittency) in a rigged waterbag. (top) Black curves are R vs. t, showing the oscillations are damped during $t\in[100,200]$. The red curve is the orbit (chaotic-regular-chaotic). (bottom) The auxiliary function ln|f(t)| vs. t with obvious demarcation of the orbital transitions.

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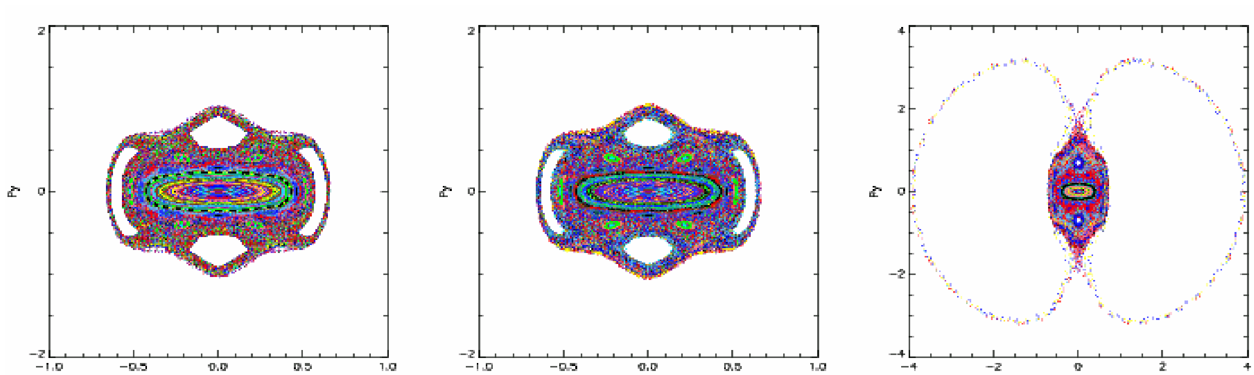

Figure 3. Poincaré sections for a waterbag with η=0.25, M=0.50, and (left-to-right) noises of $\langle|\delta\omega|\rangle$ =0, 0.0005, and 0.01.

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