TESTS OF A 3D SELF MAGNETIC FIELD SOLVER IN THE FINITE ELEMENT GUN CODE MICHELLE*

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Abstract

We present some tests of a new prototype three-dimensional (3d) self magnetic field solver in the finite-element gun code MICHELLE. The first test is a fixed ray of current in a square drift tube. The magnetic field converges linearly or quadratically with element size when using a linear or quadratic basis, respectively. The second test is a relativistic axisymmetric laminar beam expanding freely in a round drift tube. The self-consistent solution for the particle trajectories is accurate. The third test transports a beam in a uniform external magnetic field. There is no indication of beam filamentation. The computational costs for the self magnetic field solver are higher than for the Poisson solver, but the costs are still acceptable in many applications.

INTRODUCTION

We have recently implemented a prototype 3d self magnetic field solver in the finite-element gun code MICHELLE [1][2]. The new solver employs edge basis functions [3] in the curl-curl formulation of the finite-element method to compute the magnetic vector potential on unstructured grids. A novel current accumulation algorithm [4] takes advantage of the unstructured grid particle tracker to produce a compatible source vector, with which the singular matrix equation is easily solved by the conjugate gradient (CG) method.

The prototype code employs the incomplete set of linear and quadratic edge basis functions on tetrahedral, hexahedral, prism and pyramid elements, although the quadratic basis on pyramid elements needs further development. The code only works with the unstructured grid particle tracker at this time. It does not yet work with MICHELLE's Boris-like structured grid particle tracker.

The boundary conditions, $\hat{\mathbf{n}}\times\mathbf{A} = \mathbf{0}$, are appropriate for short pulsed beams where the surface currents and magnetic fields have no time to significantly diffuse through conductors. The normal component of the self magnetic field will be zero ($\hat{\mathbf{n}}\cdot\mathbf{B} = \mathbf{0}$) on conducting boundaries.

We describe three test cases below. The first case checks the field solver's computation of the magnetic vector potential $\mathbf{A}$ and the corresponding magnetic field $\mathbf{B}$ given a fixed current source. The second and third cases check the accuracy of particle trajectories in the self consistent solution, with the latter case just checking for beam filamentation. We conclude with some brief comments on computational costs.

A FIELD SOLVER TEST: ONE RAY IN A SQUARE DRIFT TUBE

The first test case is a fixed line of current (a ray) parallel to the z-axis of a square drift tube: $(x, y) \in [0, 1] \times [0, 1]$. An analytic solution for this essentially two-dimensional (2d) problem is obtained with conformal mapping. A Green's function is shown in Fig. 1.

The model geometry for the first series of tests is a cube meshed with all four 3d element types. All six faces of the cube are conductors. The elements are rotated, flipped and reordered to check that the vector potential and magnetic field are invariant to these symmetry operations. This verifies the implementation in one important way.

The model geometry for the second series of tests is one quarter of the unit square: $(x, y) \in [0, 1/2] \times [0, 1/2]$. There is one hexahedral cell in the z direction with periodic boundary conditions applied in z: $z \in [0, h]$ where $h = 1/2n$ is the element size and n is the number of hexahedral cells or edges along the model's x or y axis. The hexahedral cells are busted into a regular mesh of the various element types as desired. A four-fold rotational periodicity about the center $(x, y) = (1/2, 1/2)$ completes the model. The single ray of current passes along the center.

The magnetic field converges linearly or quadratically with element size with linear or quadratic basis functions, respectively. The magnetic vector potential also converges to our analytic solution for this model, but this is not a requirement as the curl-curl formation we employ does not fix a particular gauge for the potential.

Fig. 2 shows the convergence of B_y at the observation point $(x, y) = (1/4, 1/2)$ when using a linear basis on a mesh of tetrahedral elements. The observation point is on the edge or vertex of a number of tetrahedra, in which the field at the observation point differs. All of these field values are plotted in Fig. 2, with the solid curves connecting values from tetrahedra similarly oriented with respect to the observation point. Smooth convergence curves are thus obtained.

The field values in the tetrahedra adjacent to the observation point can be averaged, yielding the two dashed curves in Fig. 2. The interpolation errors partially cancel and yield second order convergence. Such partial cancellation occurs with the linear basis but not with the quadratic basis.

Fig. 3 shows the convergence of $\mathbf{B}$ at an arbitrarily chosen observation point. The observation point is not on the

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$$G(x,y,x',y') = \frac{1}{2\pi}\mathrm{Re}\ \ln\left(\frac{\vartheta_1\left(\frac{\pi}{2}((x+x')+i(y-y')),e^{-\pi}\right)\vartheta_1\left(\frac{\pi}{2}((x-x')+i(y+y')),e^{-\pi}\right)}{\vartheta_1\left(\frac{\pi}{2}((x-x')+i(y-y')),e^{-\pi}\right)\vartheta_1\left(\frac{\pi}{2}((x+x')+i(y+y')),e^{-\pi}\right)}\right) \tag{1}$$

Figure 1: Normalized Green's function for the magnetic vector potential on the unit square, where ϑ_1 is the first Jacobi elliptic theta function. Source: *Mathematica* function gallery.

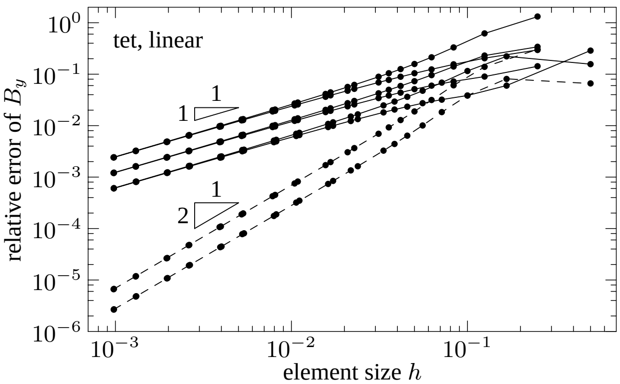

Figure 2: With a linear basis on a regular mesh of tetrahedral elements, the magnetic field component $B_y(1/4,1/2)$ converges linearly with element size. The text explains the curves.

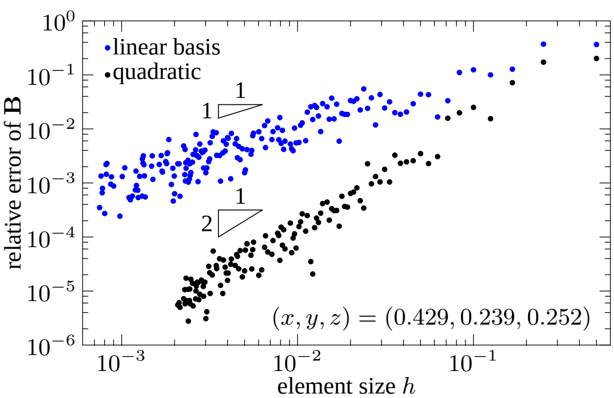

Figure 3: With a linear basis on a regular mesh of tetrahedral elements, the magnetic field $\mathbf{B}$ at an arbitrarily chosen point converges linearly with element size. With a quadratic basis, the field converges quadratically.

boundary of a tetrahedral element. The convergence is linear or quadratic depending on the basis used, but the error is not a smooth function of element size as the observation point appears at pseudo-random locations within a tetrahedral element. One infers a linear or quadratic error bound in this case.

A SELF CONSISTENT TEST: EXPANDING RELATIVISTIC BEAM

To test the self-consistent solution found by the gun code, we let an axisymmetric relativistic laminar electron beam expand freely in a round drift tube. A simple theory [5] is good for small generalized perveance K. We drift a 1 m initial radius beam for 10 m or 20 m inside a 2 m radius drift tube. The beam at its waist is injected at one end of the drift tube. The injected beam's transverse particle spacing is $h/5$, where h is the nominal element size. A 10 m example is shown in Fig. 4.

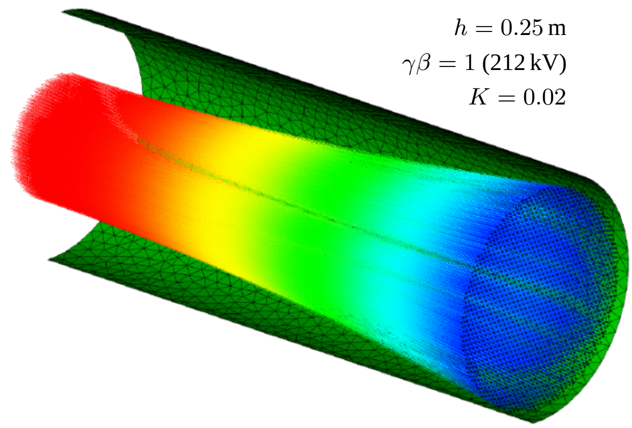

Figure 4: An axisymmetric relativistic laminar beam expands freely for 10 m in a round drift tube. Particles are colored by energy.

The agreement with the simple theory is good. Fig. 5 is a 10 m simulation showing the max and rms beam radius increases in agreement with theory. Fig. 6 is a set of 20 m simulations that show the individual particle trajectories converge close to the approximate theoretical result. The greatest error is near the beam edge, where field smoothing happens to reduce the electric and magnetic fields acting on the particles. This error gets smaller as the mesh is refined.

The gun code converges well to the self consistent solution, taking no more cycles than for a similar perveance nonrelativistic beam transport problem solved with only Poisson's equation.

A BEAM FILAMENTATION TEST: BRILLOUIN BEAM TRANSPORT

We check for beam filamentation [6] by transporting the above beam in an external magnetic field under nominally Brillouin flow conditions. We have yet to observe any beam filamentation transporting the 1 m radius beam for 20 m at various beam energies and currents. An example is shown in Fig. 7. The code still converges well.

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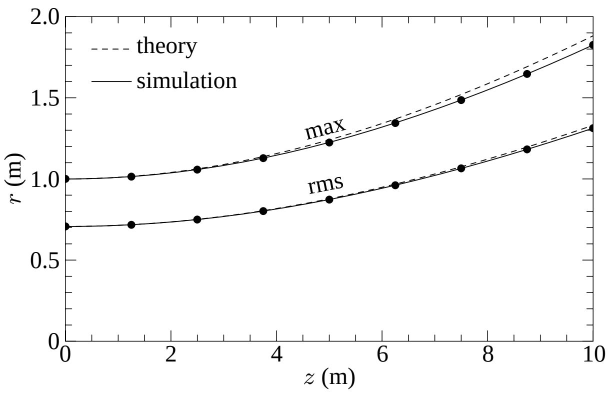

Figure 5: The simulation's max and rms beam radius agrees with the simple theory for an axisymmetric laminar freely expanding beam. The generalized perveance is $K = 0.02$. The simulation's nominal element size is $h = 0.125$ m.

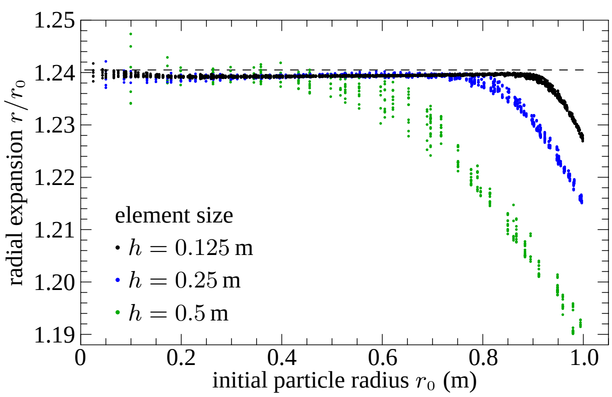

Figure 6: For three meshes, the radial expansion factor at $z = 10$ m for individual rays is plotted against initial radius. The plots show that the greatest error is near the beam edge, but the expansion does converge close to the approximate theoretical result (dashed line) as the mesh is refined. The generalized perveance is $K = 0.005$.

COMPUTATIONAL PERFORMANCE

Self magnetic field requires additional memory and time. Accumulating current and including self magnetic field in the particle tracker adds 10% to the particle tracking time. The finite element matrix and vectors are a factor of 8 larger than those for Poisson's equation. In a 2 GB process, MICHELLE can normally handle a 5M tetrahedral cell mesh but with self magnetic field it can only handle 2M tetrahedral cells. The time spent in the unpreconditioned conjugate gradient (CG) solver for self magnetic field is a factor of 11–35 (depending on the problem size relative to memory cache) over that for the diagonally preconditioned Poisson solve. This factor will also be larger if the mesh quality is poor.

As an example, the beam transport model above with element size $h = 0.25$ m has 228K tetrahedra and 1459 particles. Running 45 cycles with relative CG solver tolerances of 10^{-10} takes 20 minutes on a three year old PC:

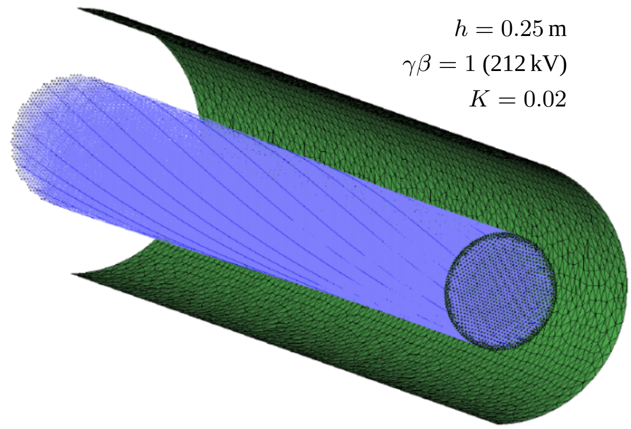

Figure 7: A foreshortened view of a 1 m radius beam transported 20 m without beam filamentation. The darker shaded particles trace the beam rotation.

a Dell Precision 530 with a 2.4 GHz Intel Xeon processor and 400 MHz RDRAM. Running 20 cycles with relative CG solver tolerances 10^{-5} gives the same result to 5 digits in 9.4 minutes.

CONCLUSION

Tests have demonstrated that the new prototype 3d self magnetic field solver obtains accurate self consistent solutions with reasonable computational costs. This encourages us to further develop the 3d solver, and also to formulate and develop a corresponding 2d solver. The latter solver will differ significantly from the current counting techniques employed by most gun codes.

REFERENCES

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